Finite Gauge Theory on Fuzzy $\mathbb{C}P^2$

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Abstract

We give a non-perturbative definition of $U(n)$ gauge theory on fuzzy $\mathbb{C}P^2$ as a multi-matrix model. The degrees of freedom are 8 hermitian matrices of finite size, 4 of which are tangential gauge fields and 4 are auxiliary variables. The model depends on a noncommutativity parameter $\frac{1}{N}$, and reduces to the usual $U(n)$ Yang-Mills action on the 4-dimensional classical $\mathbb{C}P^2$ in the limit $N \to \infty$. We explicitly find the monopole solutions, and also certain $U(2)$ instanton solutions for finite N. The quantization of the model is defined in terms of a path integral, which is manifestly finite. An alternative formulation with constraints is also given, and a scaling limit as $\mathbb{R}_\theta^4$ is discussed.

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Contents

1 Introduction

Fuzzy spaces are a nice class of noncommutative spaces, which admit only finitely many degrees of freedom, but are compatible with the classical symmetries. This means that field theory on fuzzy spaces is regularized, but compatible with a geometrical symmetry group unlike lattice field theory. A large family of such spaces is given by the quantization of (co)adjoint orbits $\mathcal{O}$ of a Lie group in terms of certain finite matrix algebras $\mathcal{O}_N$. They are labeled by a a noncommutativity parameter $\frac{1}{N}$, and the classical space is recovered in the large N limit. The simplest example is the fuzzy sphere S^2_N, which has been studied in great detail; see e.g. [1–12] and references therein. The purpose of this paper is to find a useful formulation of gauge theory on the 4-dimensional fuzzy space $\mathbb{C}P^2_N$, and to study some of its properties including topologically nontrivial solutions.

The most obvious 4-dimensional fuzzy spaces are $S^2_N \times S^2_N$ and $\mathbb{C}P^2_N$. While the former is technically easier to handle, $\mathbb{C}P^2_N$ (see e.g. [13–17]) has an 8-dimensional symmetry group $SU(3)$, which is larger than that of $S^2_N \times S^2_N$. This leads to the hope that more can be done on $\mathbb{C}P^2_N$. We propose a definition of gauge theory on $\mathbb{C}P^2_N$ in terms of certain multi-matrix models, generalizing the approach of [11]. Our basic requirement is that it should reduce to the usual $U(n)$ Yang-Mills gauge theory on classical $\mathbb{C}P^2$ in the commutative limit, but it should also be simple and promise advantages over the commutative case.

It is a well-known and fascinating fact that gauge theory on noncommutative spaces can be formulated in terms of multi-matrix models. Such matrix models also arise in string theory, e.g. the IKKT matrix model [18] and effective actions for certain D-branes [19]. This leads to a picture where the space (a "brane") arises dynamically as solution of such an action, and can be interpreted as submanifold of a higher-dimensional space. The gauge fields arise as fluctuations of the tangential coordinates, and the transversal coordinates become additional scalar fields on the brane. While the matrices are usually infinite-dimensional, they are finite-dimensional on fuzzy spaces.

For the fuzzy sphere, a formulation of gauge theory as matrix model was first given in [7], and a model which reduces to the classical Yang-Mills theory on S^2 in the large N limit was studied in detail and quantized in [11]. The formulation as matrix model has at least 2 notable features, which are not present in other formulations or in the classical case: First, it leads to a very simple picture of nontrivial gauge sectors such as monopoles, which arise as nontrivial solutions in the matrix configuration space. This was noted in [20] and fully explored in [11] for the fuzzy sphere; see also [21] for related work. The concepts of fiber bundles are not required but arise automatically, in an intrinsically noncommutative way. Second, it allows a nonperturbative quantization in terms of a finite "path" integral, which in the case of $U(n)$ Yang-Mills on S_N^2 can be carried out explicitly in the large N limit [11]. We want to see if these features can be extended to $\mathbb{C}P_N^2$. It turns out that we can indeed find monopole and (generalized) instanton solutions on $\mathbb{C}P_N^2$, generalizing the approach of [11].

Because the paper is rather long, we briefly outline the main steps here. We start with a detailed review of classical and fuzzy $\mathbb{C}P^2$ in Sections 2 and 3, including some aspects of (co)homology and differential forms. We also give a useful new link between the fuzzy and the classical differential calculus (72). Furthermore, we develop some tools in order to reformulate the usual tangential tensor fields in terms of $su(3)$ tensor fields, which arise through the embedding $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^8 \cong su(3)$. These tools are certain linear and nonlinear $su(3)$-equivariant tensor maps, which are discussed in Section 2.1 for the commutative case, and in Section 5.2 in the noncommutative case. We also show in Section 3.1 how the canonical quantum plane $\mathbb{R}_\theta^4$ can be obtained from fuzzy $\mathbb{C}P^2$ in a scaling limit.

Using this background and following [11], we propose the following action for $U(n)$ gauge theory on $\mathbb{C}P_N^2$ in Section 5:

$$S = \frac{1}{g} \int tr F_{ab} F_{ab} + S_D. \tag{1}$$

Here $F_{ab} = i[C_a, C_b] + \frac{1}{2} f_{abc} C_c$ is the field strength, and S_D is a Casimir-type constraint term (115). The basic variables are 8 "covariant coordinates" C_a, which describe the embedding of $\mathbb{C}P^2 \hookrightarrow \mathbb{R}^8$. Expanding them around the "vacuum" solution $C_a = \xi_a + A_a$ then leads to a gauge theory on $\mathbb{C}P_N^2$. The crucial point is the addition of the constraint term S_D, which is chosen such that only configurations close to the vacuum solution $C_a = \xi_a$ which defines fuzzy $\mathbb{C}P^2$ are relevant. In other words, the constraint term S_D

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stabilizes the space $\mathbb{C}P^2 \subset \mathbb{R}^8$ by giving the transversal fluctuations a large mass. The nontrivial part is to make sure that 4 physical, tangential gauge fields survive, and that the standard 4-dimensional $U(n)$ Yang-Mills theory emerges in the commutative limit. The remaining 4 degrees of freedom become very massive scalars and decouple. This is somewhat subtle, and discussed in detail in Sections 5 and 5.4. An alternative formulation of a gauge theory on $\mathbb{C}P_N^2$ is given in Section 5.5, imposing a suitable constraint which admit only tangential fields.

A nontrivial test of the proposed models is to see whether they admit topologically nontrivial solutions such as instantons. On classical $\mathbb{C}P^2$, there exist both $U(1)$ monopoles as well as $SU(2)$ instantons. The $U(1)$ monopoles are labeled by an integer which corresponds to the first Chern class, and have a selfdual field strength. Such monopoles were constructed recently on fuzzy $\mathbb{C}P^2$ in [22] as projective modules. In our formulation, it is quite straightforward to recover them as exact solutions of the equations of motions, similar as in [11]. In the commutative limit they become connections on the monopole bundles of $\mathbb{C}P^2$, which we give explicitly. In particular, we reproduce the results of [22] without having to introduce additional structure such as projective modules. In fact, all these monopole (and instanton) solutions arise in the same configuration space. This is a remarkable simplification over the commutative case, and provides further support for this approach. Similarly, we find exact solutions for the nonabelian $U(2)$ case in Section 6.2, which in the commutative limit describe certain nontrivial rank 2 "instantons" on $\mathbb{C}P^2$. The classical bundle structure is clarified in Section 6.2.3, and the $U(2)$ connection is computed explicitly in Appendix E using the Gelfand-Tsetlin basis for $su(3)$ (this takes up much of the space in the appendix). However, our purely group-theoretical ansatz only gives non-localized instantons, whose fields strength is essentially constant. Finding localized instantons and their moduli space [29] remains an interesting challenge. In particular, our "instantons" contain a nontrivial $U(1)$ component, and are neither selfdual nor anti-selfdual. The $U(1)$ seems to be related with the spin^c structure on $\mathbb{C}P^2$.

The quantization of this gauge theory is straightforward in principle, in terms of a "path integral" which is convergent. As opposed to the 2-dimensional case [11], it can no longer be performed analytically. However, we point out an interesting 2-matrix model (156) which is in the class of the models discussed above, and which might be accessible to similar analytical studies.

The presentation in this paper is detailed rather than short, but the mathematical formalism is kept at a minimum. We try to motivate the various choices made, discuss alternatives, and explain how we arrive at our models. This is sometimes done at the expense of space, but we hope that the amount of results and details justify the length of this paper.

2   Classical $\mathbb{C}P^2$

For our purpose, the most useful description of $\mathbb{C}P^2$ is as a (co)adjoint orbit in $su(3)$. In general, they have the form

$$\mathcal{O}(t) = \{gtg^{-1}; \quad g \in G\} \tag{2}$$

for some $t \in \mathfrak{g} = T_e G$. These conjugacy classes are invariant under the adjoint action of G. Then $\mathcal{O}(t)$ can be viewed as a homogeneous space:

$$\mathcal{O}(t) \cong G/K_t \tag{3}$$

where $K_t = \{g \in G : \ Ad_g(t) = 0\}$ is the stabilizer of t. The dimension and the type of $\mathcal{O}(t)$ depends only on K_t. For $G = SU(n)$, we can assume that t is a diagonal matrix. In particular, in order to obtain $\mathbb{C}P^2 = SU(3)/(SU(2) \times U(1))$ we choose

$$t = \tau_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

Here τ_a are the "conjugated" Gell-mann matrices[1] of $su(3)$, which satisfy

$$\begin{aligned} tr(\tau_a \tau_b) &= 2\delta_{ab}, \\ \tau_a \tau_b &= \frac{2}{3}\delta_{ab} + \frac{1}{2}(if_{ab}{}^c + d_{ab}{}^c)\tau_c. \end{aligned} \tag{5}$$

They are given explicitly in Appendix A, along with the tensors f_{abc} and d_{abc}. One can use (3) to derive the decomposition of the space of functions on $\mathbb{C}P^2$ into harmonics i.e. irreps under the adjoint action of $SU(3)$ [14, 22],

$$\mathcal{C}^\infty(\mathbb{C}P^2) = \bigoplus_{p=0}^{\infty} V_{(p,p)}. \tag{6}$$

Here $V_{(n,m)}$ denotes the irrep of $su(3)$ with highest weight $n\Lambda_1 + m\Lambda_2$, and Λ_i are the fundamental weights of $su(3)$.

It is convenient to work with the over-complete set of global coordinates defined by the embedding $\mathbb{C}P^2 \subset \mathbb{R}^8$ in the Lie algebra $su(3) \cong \mathbb{R}^8$. We can then write any element $X \in \mathbb{C}P^2$ as

$$\mathbb{C}P^2 = \{X = x_a \tau_a = g^{-1}tg; \ \ t = \tau_8, \ g \in SU(3)\}. \tag{7}$$

It is characterized by the characteristic (matrix) equation

$$XX = \frac{1}{\sqrt{3}} X + \frac{2}{3} \tag{8}$$

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[1]the reason for using τ_a instead of the standard Gell-mann matrices is simply that we would like to have a north pole rather than a south pole. The conventions here are different from [22].

which is easy to check for $X = \tau_8$. In component notation, this takes the form [15]

$$g^{ab} x_a x_b = 1, \tag{9}$$

$$d^{ab}_c x_a x_b = \frac{2}{\sqrt{3}} x_c. \tag{10}$$

This can be understood as follows: The matrix

$$P = \frac{1}{\sqrt{3}}(X + \frac{1}{\sqrt{3}}) \tag{11}$$

satisfies

$$P^2 = P, \qquad Tr(P) = 1 \tag{12}$$

as a consequence of (8), hence P is a projector of rank 1. Such projectors are equivalent via $P = |z_i\rangle\langle z_i|$ to complex lines in $\mathbb{C}^3$, which leads to the more familiar definition of $\mathbb{C}P^2$. Sometimes a radius R will be introduced by rescaling $x_a \to x_a R$.

Some geometry. Notice that the symmetry group $SU(3)$ contains both "rotations" as well as "translations". The generators L_a act on an element $X = x_a \tau_a \in \mathbb{C}P^2$ as

$$L_a X = [\tau_a, X] = x_b[\tau_a, \tau_b] = i f_{abc}\, x_b \tau_c. \tag{13}$$

In terms of the coordinate functions on the embedding space $\mathbb{R}^8$, this can be realized as differential operator

$$L_a = \frac{i}{2} f_{abc}(x_b \partial_c - x_c \partial_b). \tag{14}$$

Now we can identify the rotations: consider the *"north pole"*

$$X_{np} = \tau_8 = x_a \tau_a \quad \in \mathbb{C}P^2 \qquad \text{with } x_a = \delta_{a,8}.$$

The rotation subgroup is its stabilizer subalgebra $\mathfrak{r} \cong su(2) \times u(1) \subset su(3)$ generated by the "rotation" generators

$$\mathfrak{r} = \{\tau_1, \tau_2, \tau_3, \tau_8\} \tag{15}$$

resp. the corresponding[2] $L_{\mathfrak{r}}$. $\mathfrak{r}$ is clearly a subalgebra of the Euclidean rotation algebra $so(4) = su(2)_L \times su(2)_R$. The translations of X_{np} are generated by the "translation generators"

$$\mathfrak{t} = \{\tau_4, \tau_5, \tau_6, \tau_7\}. \tag{16}$$

$\mathfrak{t}$ is not a subalgebra of $su(3)$, but the following relations hold[3]:

$$[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, \qquad [\mathfrak{r}, \mathfrak{t}] = \mathfrak{t}, \qquad [\mathfrak{t}, \mathfrak{t}] = \mathfrak{r}. \tag{17}$$

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[2] we will sometimes denote the indices $1, 2, 3, 8$ with $\mathfrak{r}$, etc

[3] notice that the symmetric product defined by the d symbol respects the same subspace structure.

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It is instructive to work out these generators in terms of differential operators at the north pole $x_a = R\delta_{a,8}$ (introducing a radius R of $\mathbb{C}P^2$):

$$L_{\mathfrak{t}} = iRf_{\mathfrak{t}8\mathfrak{t}'}\partial_{\mathfrak{t}'} \tag{18}$$

Hence

$$L_4 = -\sqrt{3}iR\partial_5, \quad L_5 = \sqrt{3}iR\partial_4, \quad L_6 = -\sqrt{3}iR\partial_7, \quad L_7 = \sqrt{3}iR\partial_6 \tag{19}$$

indeed generate the 4 translations at the north pole.

There is another interesting subspace of $\mathbb{C}P^2$: the "south sphere" S_S^2 [22] or "sphere at infinity". This is a nontrivial cycle of $\mathbb{C}P^2$ which will play an important role later. Consider again the parametrization of $\mathbb{C}P^2$ in terms of 3×3 matrices $X = U^{-1}\tau_8 U$ introduced above. Using a suitable $U \in SU(3)$, we can put it into the form

$$X = \left(\begin{array}{c|c} -\frac{1}{\sqrt{3}} & 0 \\ \hline 0 & \frac{1}{2\sqrt{3}} + x_i\sigma^i \end{array}\right) = -\frac{1}{2}\tau_8 + \sum_{a=1,2,3} x_a\tau_a. \tag{20}$$

This is the subspace of $\mathbb{C}P^2$ with the most negative value of $x_8 = -\frac{1}{2}$, where $x_{4,5,6,7} = 0$ and $x_1^2 + x_2^2 + x_3^2 = \frac{3}{4}$. Hence this is a sphere of radius $\frac{\sqrt{3}}{2}$.

It is also quite illuminating to write down explicitly (10) for $i = 1, 2, 3, 8$. Using (214) and (9), they can be written as

$$\begin{aligned} \frac{1}{\sqrt{3}}(x_1 + ix_2) &= \frac{-1}{2x_8 + 1}(x_4 + ix_5)(x_6 - ix_7), \\ \frac{2}{\sqrt{3}}x_3 &= \frac{1}{2x_8 + 1}(x_6^2 + x_7^2 - x_4^2 - x_5^2), \\ (1 - x_8)(1 + 2x_8) &= \frac{3}{2}(x_4^2 + x_5^2 + x_6^2 + x_7^2). \end{aligned} \tag{21}$$

This shows how the "transversal" variables $x_{\mathfrak{r}}$ are expressed in terms of the tangential ones. The other equations of (10) are redundand except at the south sphere. In particular, note that fixing a value for x_8 determines a 3-sphere $S^3 \subset \mathbb{R}_{\mathfrak{t}}^4 = \langle x_4, x_5, x_6, x_7 \rangle$. Then the first 2 equations above determine a map $S^3 \to S^2$, which is precisely the Hopf map. This shows explicitly that $\mathbb{C}P^2$ is a compactification of $\mathbb{R}^4$ via a sphere "at infinity".

2.1 $SU(3)$ tensors and constraints

In order to better understand the fuzzy case, it is useful to consider tensor fields on $\mathbb{C}P^2$ with indices in the adjoint of $su(3)$; these will be denoted as "tensors" throughout this paper. Their transformation under the symmetry group $SU(3)$ is defined in the obvious way. Examples are the above global "coordinates" x_a, the field strength $F_{ab}(x)$ defined below etc. The best way to think of them is as pull-backs via the embedding

map $\mathbb{C}P^2 \subset \mathbb{R}^8$ of ordinary tensors which live on $\mathbb{R}^8$. The "rank" of such tensors will be the number of $su(3)$ indices. In order to relate them to the usual tangential tensor fields, we introduce some useful maps which separate the tangential from the transversal components. See also [15, 16] for similar considerations.

2.1.1 Linear tensor maps

For rank one tensors (one-forms) $X = X_a dx_a \cong X_a \tau_a$ in the above sense, consider the map [16]

$$J(X) := -\frac{i}{\sqrt{3}} [x^a \tau_a, X]. \tag{22}$$

In component form, it is a linear map of tensors

$$J(X)_a = \frac{1}{\sqrt{3}} f_{abc} x_b X_c. \tag{23}$$

It is easy to see (e.g. at the north pole) that it takes the eigenvalue 0 on non-tangential fields, and rotates the tangential ones with $J^2 = -1$ on tangential fields. J corresponds to the complex structure on $\mathbb{C}P^2$ [15].

Similarly, consider the map[4]

$$D^{lin}(X)_a = \frac{1}{\sqrt{3}} d_{abc} x_b X_c - \frac{1}{3} X_a \tag{24}$$

for any tensor X. At the north pole, it is

$$D^{lin}(X)_a = \frac{1}{\sqrt{3}} d_{a8c} X_c - \frac{1}{3} X_a \tag{25}$$

which has the eigenvalues -1 for $a = 1, 2, 3$, and $+\frac{1}{3}$ for $a = 8$, and 0 for $a = 4, 5, 6, 7$. Therefore the space of **tangential tensors** is globally characterized by

$$\boxed{D^{lin}(X)_a = 0.} \tag{26}$$

Finally, the map

$$P(X)_a := x_a(x_b X_b) \tag{27}$$

projects on the "radial" part of X, and has eigenvalues $0, 1$.

Let us see explicitly how D^{lin} decomposes a tensor X_a. Using the identity (226), one finds

$$(D^{lin})^2(X)_a = -\frac{2}{3} D^{lin}(X)_a + \frac{1}{3} X_a + \frac{1}{9} J^2(X)_a \tag{28}$$

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[4]A similar operator was introduced in [16]

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Moreover, the identity

$$\sum_e (f^{ade}d^{bce} + f^{bde}d^{eca} + f^{cde}d^{eab}) = 0 \tag{29}$$

implies

$$[J, D^{lin}] = 0, \tag{30}$$

while $[P, D] = 0 = [P, J]$ is obvious. Similarly, contracting the identity

$$\sum_e f^{abe}f^{cde} = \frac{8}{3}(\delta^{ac}\delta^{bd} - \delta^{bc}\delta^{ad}) + \sum_e (d^{ace}d^{bde} - d^{ade}d^{bce}) \tag{31}$$

with $x_d x_b X_c$ gives

$$-J^2(X)_a + 3(D^{lin})^2(X)_a = -\frac{8}{3}P(X)_a + 3X_a \tag{32}$$

which together with (28) implies

$$(D^{lin})^2(X)_a + D^{lin}(X)_a = \frac{4}{9}P(X)_a \tag{33}$$

Now suppose we have a fixed point $D^{lin}(X)_a = 0$, i.e. X_a is tangential. Then (33) implies $P(X) = 0$, i.e.

$$x \cdot X = 0 \tag{34}$$

as it should, i.e. the "radial" component vanishes. Further, suppose we have a fixed point $D^{lin}(X)_a = \frac{1}{3}X_a$. Then (33) gives $X_a = P(X)_a$, hence X_a is purely radial,

$$X_a = x_a f(x). \tag{35}$$

The maps D^{lin} and P can be used to project any tensor field to its tangential components, by

$$X \to X_{tang} = X - \frac{4}{3}P(X) + D^{lin}(X). \tag{36}$$

2.1.2   Nonlinear tensor maps

We want to understand better the constraint (10). In particular, we can show that (10) implies (9). To see this, define the following nonlinear tensor map

$$D^{nl}(X)_a = d_{abc}X_b X_c - \frac{2}{\sqrt{3}}X_a. \tag{37}$$

Assume we have some "eigenvector" of D^{nl}, which we write as

$$D^{nl}(X)_a = (\alpha - \frac{2}{\sqrt{3}})X_a \tag{38}$$

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for some constant α, i.e. $d_{abc}X_bX_c = \alpha\, X_a$. By rewriting $dX(dXX)$ using (226) and $f_{abc}X_bX_c = 0$, one finds

$$\alpha^2\, X_a = \frac{4}{3}(X \cdot X)X_a \tag{39}$$

hence

$$\alpha^2 = \frac{4}{3}(X \cdot X). \tag{40}$$

In particular, it follows that $X \cdot X$ is a constant, and

(9) is a consequence of (10).

Therefore a tensor field $X_a(x)$ which satisfies $D^{nl}(X) = 0$ describes functions $X : \mathbb{C}P^2 \to \mathbb{C}P^2$, with normalization $X \cdot X = 1$. Furthermore, note that for infinitesimal variations δX we have $\delta D^{nl}(X) = 2\sqrt{3}\, D^{lin}(\delta X)$. This means that the linear constraint $D^{lin}(\delta X) = 0$ describes tangential fields on $\mathbb{C}P^2$, in agreement with the previous findings.

In the noncommutative case, we will see that a fuzzy version of X_a satisfying (approximately) $d_{abc}X_bX_c = \frac{2}{\sqrt{3}}X_a$ admits 4 tangential degrees of freedom, which are identified as gauge fields. Hence gauge theory can be interpreted as a theory of fluctuating (covariant, fuzzy) coordinates of $\mathbb{C}P^2$.

2.2   Symplectic form and (anti-) selfduality

$\mathbb{C}P^2$ is a symplectic (Kähler) space. The symplectic (Kähler) form is given by[5]

$$\eta = \frac{1}{2\sqrt{3}R}f_{abc}x_a dx_b dx_c \tag{41}$$

which is clearly invariant under $SU(3)$. Here η is normalized such that $\langle \eta, \eta \rangle = 2$ where $\langle , \rangle$ is the obvious inner product for forms. The volume form is then given by $dV = \frac{1}{2}\eta^2$. In particular, it follows that η is *selfdual*: By $su(3)$ invariance it suffices to check this at the north pole $x_a = \delta_{a,8}$. There f_{ab8} is manifestly selfdual, and so is $\eta f(x)$ for any function $f(x)$.

Furthermore, we claim that the 2-forms of the form

$$\alpha_2 = f_{abc}dx_a dx_b A_c(x) \quad \text{with} \quad \langle \alpha_2, \eta \rangle = 0 \tag{42}$$

span the space of anti-selfdual 2-forms. To see this, note again that the space of such 2-forms is invariant under $SU(3)$. It therefore suffices again to consider the north pole, where due to the explicit form of f_{abc} the space is easily seen to be 3-dimensional and anti-selfdual. This will be useful in the context of instantons.

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[5]to see that it is closed, note that $d\eta \propto f_{abc}dx_a dx_b dx_c = 0$ on $\mathbb{C}P^2$ due to the explicit form of f_{abc}

3 Fuzzy $\mathbb{C}P^2_N$.

We start by recalling the definition of fuzzy $\mathbb{C}P^2$ [14, 15], in order to fix our conventions. In general, (co)adjoint orbits (2) on G can be quantized (see [23], and [24] for an alternative approach for matrix Lie groups G) in terms of a simple matrix algebra $Hom_{\mathbb{C}}(V_N)$, where V_N are suitable representations of G. The appropriate representations V_N can be identified by matching the spaces of harmonics, i.e. the decomposition into irreps under the symmetry group G [24]. For $\mathbb{C}P^2$, the correct harmonics are reproduced for[6] $V_N = V_{N\Lambda_2} = V_{(0,N)}$, which is the irrep of $su(3)$ with highest weight $N\Lambda_2$. Then the space of "functions" on fuzzy $\mathbb{C}P^2$ decomposes as

$$\mathbb{C}P^2_N := Mat(V_{(0,N)}) = V_{(0,N)} \otimes V^*_{(0,N)} = \oplus^N_{n=0} V_{(n,n)}. \tag{43}$$

under the (adjoint) action of $SU(3)$. This matches the decomposition (6) of functions on $\mathbb{C}P^2$ up to the cutoff. To identify the fuzzy coordinate functions, we denote with

$$\xi_a = \pi_{V_N}(T_a) \tag{44}$$

the representation of the generators T_a (209) of the Lie algebra $su(3)$ acting on $V_{N\Lambda_2}$. Now consider the $3D_N \times 3D_N$ matrix (D_N is the dimension (55) of $V_{N\Lambda_2}$)

$$X = \sum_a \xi_a \tau^a \tag{45}$$

where τ_a are the conjugated Gell-Mann matrices. As shown in Appendix B (223), X satisfies the characteristic equation

$$(X - \frac{2N}{3})(X + \frac{N}{3} + 1) = 0. \tag{46}$$

On the other hand, (5) implies

$$X^2 = \xi_a \xi_b \, (\frac{2}{3}\delta^{ab} + \frac{1}{2}(if^{ab}{}_c + d^{ab}{}_c)\tau^c). \tag{47}$$

Together with (46) and the fact that ξ_a are representations of $su(3)$, this gives the coordinate form of the algebra of functions on $\mathbb{C}P^2_N$,

$$if^{ab}_c \xi_a \xi_b = -3\, \xi_c, \qquad [\xi_a, \xi_b] = \frac{i}{2} f^{ab}_c \xi^c \tag{48}$$

$$g^{ab}\xi_a\xi_b = \frac{1}{3}N^2 + N, \tag{49}$$

$$d^{ab}_c \xi_a \xi_b = (\frac{2N}{3} + 1)\, \xi_c. \tag{50}$$

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[6]alternatively one could use $V_{N\Lambda_1} = V_{(N,0)}$, which gives an equivalent algebra but a different embedding of $\mathbb{C}P^2 \subset \mathbb{R}^8$. Our choice matches the classical geometry in Section 2.

E.g. for $N = 1$, we simply have $\xi_a = \frac{1}{2}\tau_a$. One then defines the rescaled variables $x_i = (x_1, ...x_8)$ of $\mathbb{C}P_N^2$ as

$$x_a = \Lambda_N \, \xi_a, \qquad \Lambda_N = R\frac{1}{\sqrt{\frac{1}{3}N^2 + N}} \tag{51}$$

which satisfy the relations [15]

$$if_c^{ab}x_ax_b \;=\; -3\Lambda_N \, x_c = -3\frac{R}{\sqrt{\frac{1}{3}N^2 + N}} \, x_c, \tag{52}$$

$$g^{ab}x_ax_b \;=\; R^2, \tag{53}$$

$$d_c^{ab}x_ax_b \;=\; R \, \frac{2N/3 + 1}{\sqrt{\frac{1}{3}N^2 + N}} \, x_c. \tag{54}$$

Here R is an arbitrary radius, which will usually be 1 in this paper. Furthermore,

$$D_N := dim(V_{(0,N)}) = (N+1)(N+2)/2 \tag{55}$$

from Weyl's dimension formula, hence the algebra of functions on fuzzy $\mathbb{C}P_N^2$ is simply $Mat(D_N, \mathbb{C})$.

The decomposition (43) of functions into harmonics defines an embedding of the spaces

$$\mathbb{C}P_N^2 \hookrightarrow \mathbb{C}P_{N+1}^2 \hookrightarrow ... \hookrightarrow \mathcal{C}^\infty(\mathbb{C}P^2) \tag{56}$$

(not as algebras), by matching the harmonics of $su(3)$. Moreover, there is a corresponding equivariant quantization map $T^N : \mathcal{C}^\infty(\mathbb{C}P^2) \to \mathbb{C}P_N^2$ [14] obtained by cutting off the higher modes, which makes precise how the algebras approach the classical one as $N \to \infty$. This allows to measure the "difference" between fuzzy and classical functions using the operator norm resp. supremum norm, and statements like $f \in \mathbb{C}P_N^2 \to f \in \mathcal{C}^\infty(\mathbb{C}P^2)$ as $N \to \infty$ are understood in this sense throughout this paper.

Additional structure. We can easily identify a "north pole" in the fuzzy case[7]. Indeed ξ_8 and ξ_3 can be simultaneously diagonalized, and the highest weight state $|N\Lambda_2\rangle$ of $V_{(0,N)}$ has eigenvalues $\xi_8|N\Lambda_2\rangle = \frac{N}{\sqrt{3}}|N\Lambda_2\rangle$ and $\xi_3|N\Lambda_2\rangle = 0$. This is the unique vector in $V_{(0,N)}$ with this maximal eigenvalue of ξ_8. It is therefore natural to identify the delta-function on the north pole with the projector $|N\Lambda_2\rangle\langle N\Lambda_2|$, i.e. to consider $\langle N\Lambda_2|f(x)|N\Lambda_2\rangle$ as value of $f(x) \in \mathbb{C}P_N^2$ "at the north pole". For large N, the eigenvalue of x_8 approaches R as it should.

We can similarly find a fuzzy "south sphere" corresponding to the sphere in (20). Indeed, consider the subspace of $V_{(0,N)}$ with minimal eigenvalue $-\frac{N}{2\sqrt{3}}$ of ξ_8. It is isomorphic

---

[7]for $V_{(N,0)}$ there would be a south pole and a north sphere

to a $N+1$-dimensional irrep of the $su(2)$ subalgebra generated by $\xi_{1,2,3}$, hence it is a fuzzy sphere with $\xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{N(N+2)}{4}$. Therefore $x_1^2 + x_2^2 + x_3^2 = \frac{3}{4}$ for large N as in (20).

The "angular momentum" operators (generators of $SU(3)$) become now inner,

$$L_a f(x) = [\xi_a, f], \tag{57}$$

because then $L_a x_b = [\xi_a, x_b] = \frac{i}{2} f_{ab}{}^c x_c$, as classically. The integral on $\mathbb{C}P_N^2$ is defined by the suitably normalized trace,

$$\int f(x) = \frac{1}{D_N} Tr(f) \tag{58}$$

and is invariant under $SU(3)$.

As a final remark, let us reconsider the characteristic equation (46) with eigenvalues $x = -\frac{N}{3} - 1$ resp. $x = \frac{2N}{3}$. Their multiplicities $n_\pm$ can be obtained from $Tr(X) = 0 = n_-(-\frac{N}{3} - 1) + n_+(\frac{2N}{3})$, which implies

$$n_- = N(N+2), \qquad n_+ = \frac{1}{2}(N+2)(N+3). \tag{59}$$

This motivates to introduce the projector

$$P = \frac{X + \frac{N}{3} + 2}{N+1} \tag{60}$$

which satisfies

$$P^2 = P, \qquad Tr(P) = \frac{1}{2}(N+2)(N+3) \approx D_N. \tag{61}$$

This clearly generalizes (12) to the noncommutative case.

3.1 Scaling limit: quantum plane

Consider the coordinates

$$z_a = \frac{\xi_a}{\sqrt{N}} \tag{62}$$

for large N. The commutation relations are

$$[z_a, z_b] = \frac{i}{2\sqrt{N}} f_c^{ab} z^c \tag{63}$$

$$g^{ab} z_a z_b = \frac{N}{3} + 1, \tag{64}$$

$$z_c = \frac{3}{2\sqrt{N}} \frac{1}{1 + \frac{3}{2N}} d_c^{ab} z_a z_b. \tag{65}$$

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We are interested in the neighborhood of the "north pole", where $z_8 \approx \sqrt{N/3}$ and $z_{\mathfrak{t}} = o(1)$. Then (21) can be used to solve for $z_{1,2,3}$, which implies that $z_{1,2,3} = o(\frac{z_{\mathfrak{t}}^2}{\sqrt{N}})$. Then (63) implies

$$[z_4, z_5] = -\frac{i}{2} + o(\frac{z_{\mathfrak{t}}^2}{N}), \qquad [z_6, z_7] = -\frac{i}{2} + o(\frac{z_{\mathfrak{t}}^2}{N}). \tag{66}$$

in the large N limit, we obtain $\mathbb{R}^4_\theta$ with

$$\theta_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{67}$$

4 Differential calculus

A differential calculus on $\mathbb{C}P^2$ was introduced in [22], which we recall and somewhat extend here. We introduce a basis of one-forms θ_a, $a = 1, 2, ..., 8$ à la Madore [2], which transform in the adjoint of $su(3)$ and commute with the algebra of functions:

$$[\theta_a, f] = 0, \qquad \theta_a \theta_b = -\theta_b \theta_a. \tag{68}$$

This defines a space of exterior forms on fuzzy $\mathbb{C}P^2_N$, which we denote by $\Omega^*_N := \Omega^*(\mathbb{C}P^2_N)$. The gradation given by the number of anticommuting generators θ_a. The highest non-vanishing form is the 8-form corresponding to the volume form of $SU(3)$.

One can also define an exterior derivative $d : \Omega^k_N \to \Omega^{k+1}_N$ such that $d^2 = 0$ and imposing the graded Leibniz rule. Its action on the algebra elements $f \in \Omega^0_N$ is given by the commutator with a special one-form: Consider the invariant one-form

$$\Theta = \xi_a \theta_a. \tag{69}$$

The exterior derivative of a function $f \in \mathbb{C}P^2_N$ is then given by

$$df := [\Theta, f] = [\xi_a, f]\theta_a. \tag{70}$$

In particular, we have

$$d\xi_b = [\xi_a, \xi_b]\theta_a = \frac{i}{2} f_{abc}\theta_a\xi_c. \tag{71}$$

Note that for large N, this can be expressed in terms of the map J as

$$dx_a = \frac{i\sqrt{3}}{2} J(\theta)_a. \tag{72}$$

This means that only tangential forms survive in the commutative limit, while the transversal forms becomes zero. Therefore this calculus reduces to the usual calculus on $\mathbb{C}P^2$ for large N. In particular, we define

$$\langle \theta_a, \theta_b \rangle = -\frac{4}{3}\delta_{ab} \tag{73}$$

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which for large N implies $\langle dx_a, dx_b \rangle = \delta_{ab}$. Furthermore, we define

$$\eta = -\frac{\sqrt{3}}{8} f_{abc} x_a \theta_b \theta_c \tag{74}$$

which satisfies $\langle \eta, \eta \rangle = 2$ for any N. For $N \to \infty$, it reduces to the symplectic form (41)

$$\eta \to -\frac{\sqrt{3}}{8} f_{abc} x_a J(\theta_b) J(\theta_c) = \frac{1}{2\sqrt{3}} f_{abc} x_a dx_b dx_c. \tag{75}$$

The definition of d on higher forms is straightforward, once we have $d : \Omega_N^1 \to \Omega_N^2$ such that $d^2(f) = 0$. Notice [24] first that there is a natural bimodule-map from one-forms to 2-forms, given by

$$\star_1 (\theta_a) := \frac{i}{4} f_{abc} \theta_b \theta_c. \tag{76}$$

Using this, we define

$$\begin{aligned} d : \Omega_N^1 &\to \Omega_N^2, \\ \alpha &\mapsto d\alpha = [\Theta, \alpha]_+ - \star_1(\alpha) \end{aligned} \tag{77}$$

where $\alpha \in \Omega_N^1$. One can then verify $d^2 = 0$ in general. To see this, note that

$$ddf = [\Theta, df]_+ - \star_1(df) = 0$$

using the following relation:

$$\star_1 (\Theta) = \Theta^2. \tag{78}$$

This follows from

$$\Theta^2 = \Theta\Theta = \frac{1}{2} \theta_a \theta_b [\xi_a, \xi_b] = i\frac{1}{4} f_{abc} \theta_a \theta_b \xi_c = -\frac{2i}{\sqrt{3}\Lambda_N} \eta. \tag{79}$$

One can also show that

$$d\Theta = \Theta^2, \tag{80}$$

which implies

$$d\eta = 0. \tag{81}$$

Field strength For an arbitrary one-form

$$C = C_a \theta_a \tag{82}$$

we define the field strength by

$$F := CC - \star_1 C = -\frac{i}{2} F_{ab} \theta_a \theta_b. \tag{83}$$

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Upon writing

$$C = \Theta + A \tag{84}$$

this becomes

$$F = dA + AA. \tag{85}$$

Since the differential calculus reduces to the classical one for large N, this reduces indeed to the correct expression for the field strength provided the "gauge fields" A are purely tangential. How to implement this requirement will be discussed in Section 5.2. The inner product of forms (73) extends as usual:

$$\langle \theta_a, \theta_b \rangle := -\frac{4}{3} \delta_{ab}, \qquad \langle \theta_a \theta_b, \theta_c \theta_d \rangle := \frac{16}{9} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}), \tag{86}$$

and one can define an analog of the Yang-Mills action by

$$S_{YM} = Tr\langle F, F \rangle. \tag{87}$$

One could now proceed and define the integrals of forms. Here one meets an obstacle: there is no constant invariant 4-form which could define the volume-form on fuzzy $\mathbb{C}P^2$. The most natural candidate $d^4V = \eta^2$ does not commute with functions, and is therefore not useful for gauge theory. Correspondingly, there seems to be no natural notion of a (gauge-invariant) Hodge star. This somewhat obscures the meaning of instantons in the fuzzy case, lacking the concept of a self-dual field strength. Therefore these notions only make sense in the commutative of large N limit. The construction of topological invariants suffers from the same problem. This will mean that we will calculate e.g. Chern numbers only in the commutative limit. This problem is well-known for fuzzy spaces, and perhaps not too surprising since the notion of topology for spaces defined in terms of finite algebras can only arise asymptotically. We discuss in the following paragraph how such "asymptotic" Chern numbers can be computed.

4.1   Asymptotic Chern numbers

The de Rham cohomology of classical $\mathbb{C}P^2$ is given by $H^2(\mathbb{C}P^2) = \mathbb{R}\eta$ and $H^4(\mathbb{C}P^2) = \mathbb{R}\eta^2$. The integer cohomology $H^{2*}(\mathbb{C}P^2; \mathbb{Z})$ is generated by

$$\omega = \frac{\eta}{3\pi}, \tag{88}$$

i.e.

$$\int_{S^2_S} \omega = 1 = \int_{\mathbb{C}P^2} \omega \wedge \omega \tag{89}$$

where the 2- and 4-cycles are represented by the south sphere S^2_S discussed in Section 2, and $\mathbb{C}P^2$ itself. We therefore expect 2 interesting Chern classes $c_1 = \frac{i}{2\pi} trF$ and

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$c_2 = -\frac{1}{8\pi^2}(trF \wedge trF - tr(F \wedge F))$. The first Chern number is given by the integral over the south sphere

$$c_1 = \frac{i}{2\pi} \int_{S_S^2} trF = \frac{i}{2\pi} \int_{S_S^2} \frac{1}{2} tr\langle F, \eta \rangle \tag{90}$$

since $\langle \eta, \eta \rangle = 2$. Now

$$\langle F, \eta \rangle = \frac{\sqrt{3}i}{16} F_{ab} f_{def} x_d \langle \theta_a \theta_b, \theta_e \theta_f \rangle = \frac{2i}{3\sqrt{3}} F_{ab} f_{abc} x_c = \frac{4i}{3\sqrt{3}} F_c x_c \approx \frac{4i}{3N} F_c C_c \tag{91}$$

where F_a is defined as in (99). Here "$\approx$" means equal up to $o(1/N)$. This will be evaluated below for the configurations of interest. Another way to compute c_1 which is especially interesting for the noncommutative case is to note that $\eta \wedge c_1 \in H^4(\mathbb{C}P^2) \cong \mathbb{R}$, hence

$$\begin{aligned} c_1 &= \frac{1}{3\pi} \int_{\mathbb{C}P^2} \eta \wedge c_1 = \frac{i}{6\pi^2} \int_{\mathbb{C}P^2} \eta \wedge trF \\ &= \frac{i}{6\pi^2} \int_{\mathbb{C}P^2} \star\eta \wedge trF = \frac{i}{6\pi^2} \int_{\mathbb{C}P^2} tr\langle \eta, F \rangle d^4V \\ &\approx -\frac{2}{9\pi^2 N} \int_{\mathbb{C}P^2} trF_c C_c d^4V = -\frac{1}{ND_N} TrF_c C_c \end{aligned} \tag{92}$$

for large N. The integral is now over the entire $\mathbb{C}P^2$, and $d^4V = \frac{1}{2}\eta^2$ is the volume form on $\mathbb{C}P^2$. Here we use the metric $d^2s = \sum_{a=1}^{8} dx_a dx_a$; this leads to the volume $Vol(\mathbb{C}P^2) = \int_{\mathbb{C}P^2} d^4V = \frac{9}{2}\pi^2$. We note that the last expression in (92) has a topological meaning, which will be important later as a possible term in the action of a gauge theory.

5 Multi-Matrix Models and Yang-Mills action

5.1 Degrees of freedom and field strength

Our basic degrees of freedom are 8 hermitian matrices $C_a \in Mat(D_N, \mathbb{C})$ transforming in the adjoint of $su(3)$, which are naturally arranged as a single $3D_N \times 3D_N$ matrix

$$C = C_a \tau^a + C_0 \mathbb{1} \tag{93}$$

where $C_0 = 0$ in much of the following. The size D_N (55) of these matrices will be relaxed later. We want to find a multi-matrix model in terms these C_a, which for large N reduces to (euclidean) Yang-Mills gauge theory on $\mathbb{C}P^2$. The idea is to interpret the C_a as suitably rescaled "covariant coordinates" [31] on fuzzy $\mathbb{C}P_N^2$, with the gauge transformation

$$C_a \rightarrow U^{-1} C_a U \tag{94}$$

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for unitary matrices U (of the same size). The C_a can also be interpreted as components of a one-form $C = C_a \theta_a$ as in (82). Following the approach of [11], we look for an action which has the "vacuum" solution

$$C_a = \xi_a \tag{95}$$

corresponding to $\mathbb{C}P_N^2$, and forces C_a to be "approximately" the corresponding representation $V_{N\Lambda_2}$ of $su(3)$. Then the fluctuations

$$C_a = \xi_a + A_a \tag{96}$$

are small and describe the gauge fields. By inspection, these gauge fields A_a transform as

$$\delta A_a = i[\xi_a + A_a, \Lambda] = iL_a\Lambda + i[A_a, \Lambda] \tag{97}$$

for $U = e^{i\Lambda}$, which is the appropriate formula for a gauge transformation. Since the C_a resp. ξ_a correspond to "global" coordinates in the embedding space $\mathbb{R}^8$, we can hope that nontrivial solutions such as instantons can also be described in this way.

A suitable definition for the field strength is then given by (83),

$$\boxed{F_{ab} = i[C_a, C_b] + \tfrac{1}{2}\, f_{abc}C_c = i(L_aA_b - L_bA_a + [A_a, A_b]) + \tfrac{1}{2}\, f_{abc}A_c.} \tag{98}$$

We will also need

$$\boxed{\begin{aligned} F_a &= i f_{abc}C_bC_c + 3C_a = \tfrac{1}{2}f_{abc}F_{ab}, \\ D_a &= d_c^{ab}C_aC_b - \left(\tfrac{2N}{3} + 1\right)C_c. \end{aligned}} \tag{99}$$

Under gauge transformations, the field strength transforms as

$$F_{ab} \to U^{-1}F_{ab}U. \tag{100}$$

As discussed in Section 4, this can also be interpreted as 2-form

$$F = dA + AA = \frac{2i}{3}F_{ab}J(dx_a)J(dx_b) \tag{101}$$

using (72), (83), if one considers the fields C_a as one-forms $C = C_a\theta_a = \Theta + A$. Furthermore, we will show that F_{ab} is tangential if C_a satisfies (approximately) the constraints of $\mathbb{C}P^2$. Assuming that A_a tend to well-defined functions on $\mathbb{C}P^2$ in the large N limit and using (72), this implies that F_{ab} are the components of the usual field strength 2-form in the commutative (large N) limit, transformed by J acting on the indices. This justifies the above definition of F_{ab}, and it is a matter of taste whether one works with the components or with forms. Note that our C and A are dimensionless, and the usual dimensions are recovered upon introducing a radius R and suitably rescaling the quantities $A_a \to RA'_a$ etc.

It is instructive to consider F_a explicitly: at the north pole, its components are

$$
\begin{aligned}
F_1 &= i((L_4 A_7 - L_7 A_4) - (L_5 A_6 - L_6 A_5)) \\
F_2 &= i((L_4 A_6 - L_6 A_4) + (L_5 A_7 - L_7 A_5)) \\
F_3 &= i((L_4 A_5 - L_5 A_4) - (L_6 A_7 - L_7 A_6)) \\
F_8 &= -i\sqrt{3}((L_4 A_5 - L_5 A_4) + (L_6 A_7 - L_7 A_6)).
\end{aligned} \tag{102}
$$

We see explicitly that $F_{1,2,3}$ are antiselfdual, while F_8 is selfdual.

5.2   Constraints

In order to describe fuzzy $\mathbb{C}P^2$, the fields C_a should satisfy at least approximately the constraints (49), (50) of $\mathbb{C}P_N^2$,

$$
D_a = 0, \tag{103}
$$

$$
g_{ab} C_a C_b = \frac{1}{3} N^2 + N \tag{104}
$$

which are gauge invariant. These constraints ensure that C_a can be interpreted as describing a ("dynamical" or fluctuating) $\mathbb{C}P_N^2$. However, notice that they are not independent at least in the commutative limit as was shown in Section 2.1: (10) implies (9). To understand these constraints in the noncommutative case, we introduce some analogs of the maps in Section 2.1.

5.2.1   Linear tensor maps

Given 8 matrices C_a as above, we can define the following maps of matrices X_a with index in the adjoint of $su(3)$:

$$
\begin{aligned}
\mathcal{J}(X)_a &= \frac{1}{2N} f_{abc} \{C_b, X_c\}, \\
\mathcal{D}^{lin}(X)_a &= \frac{1}{2N} d_{abc} \{C_b, X_c\} - (\frac{1}{3} + \frac{1}{2N}) X_a \\
\mathcal{P}(X)_a &= \frac{1}{\frac{4}{3} N^2 + 4N} \{C_a, \{C_b, X_b\}\}.
\end{aligned} \tag{105}
$$

They map hermitian matrices into hermitian matrices, and depend on the particular "background" field C. Note that $\mathcal{J}$ is anti-selfadjoint in the following sense

$$
Tr(\mathcal{J}(X)_a Y_a) = -Tr(X_a \mathcal{J}(Y_a)) \tag{106}
$$

while $\mathcal{D}^{lin}$ is selfadjoint. Writing $C_a = \xi_a + A_a$, these maps reduce to the corresponding maps in Section 2.1 in the commutative limit

$$
\mathcal{D}^{lin}(X)_a \to D^{lin}(X)_a, \qquad \mathcal{J}(X)_a \to J(X)_a, \qquad \mathcal{P}(X)_a \to P(X)_a \tag{107}
$$

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as $N \to \infty$ independent of C_a, provided A_a and X_a are "smooth". With smooth we mean that both X_a and its derivatives $[\xi_b, X_a]$ become smooth functions as $N \to \infty$, but mild singularities may be allowed. Therefore tangential tensors could be described by solutions of $\mathcal{D}^{lin}(X)_a = 0$. However this seems to be too restrictive in the noncommutative case, and we will only require

$$\mathcal{D}^{lin}(X)_a \to 0 \quad \text{as} \quad N \to \infty. \tag{108}$$

Recall that this is to be understood in the sense of (56). Alternatively, they can also be described by the image of $\mathcal{J}(X)$, which is equivalent for large N.

Using (107), the maps $\mathcal{D}^{lin}$ and $\mathcal{P}$ can be used to project any tensor field to its tangential components for large N as in (36), by

$$X \to X_{tang} = X - \frac{4}{3}\mathcal{P}(X) + \mathcal{D}^{lin}(X). \tag{109}$$

The resulting field X_{tang} satisfies (108), which is enough for our purpose. Therefore imposing (108) allows 4 tangential degrees of freedom in the large N limit.

5.2.2 Nonlinear tensor map

One can also consider the noncommutative versions of the map in Section 2.1.2:

$$\mathcal{D}^{nl}(C_a) \;\; = \;\; \frac{1}{N}\, D_a = \frac{1}{N}\, (d_{abc}C_b C_c - (\frac{2N}{3} + 1)C_a), \tag{110}$$

which maps hermitian matrices into hermitian matrices. For infinitesimal variations, we have $\delta\mathcal{D}^{nl}(C) = \mathcal{D}^{lin}(\delta C)$. It would be tempting to impose the constraint $\mathcal{D}^{nl}(C_a) = 0$. However, it is not clear whether this equation has enough solutions in the noncommutative case to describe 4 tangential vector fields. As this is probably not be the case, we will not pursue this strict constraint any further, and we only require

$$\mathcal{D}^{nl}(C)_a \to 0 \quad \text{as} \quad N \to \infty. \tag{111}$$

This guarantees the correct commutative limit, and it is equivalent to (108) as long as A_a is "smooth" since then $\mathcal{D}^{nl}(C)_a = \mathcal{D}^{lin}(A)_a + o(\frac{1}{N})$. Therefore (111) admits 4 tangential degrees of freedom in the large N limit via (109). Moreover, (111) holds for finite-action configurations of the Yang-Mills action defined below.

Nevertheless, there exists a slightly modified constraint which can be imposed exactly, which will be discussed in Section 5.5.

5.2.3 Constraints and field strength

We can now verify that F_{ab} is (approximately) tangential in the sense that $\mathcal{D}^{lin}(F) \to 0$ for each index as $N \to \infty$, provided both $\mathcal{D}^{nl}(C)_a \to 0$ and its derivatives $[C_a, \mathcal{D}^{nl}(C)_b] \to 0$ for $N \to \infty$.

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Generalizing the above definition to higher tensors, consider

$$
\begin{aligned}
\mathcal{D}_1^{lin}(F_{ad}) &= \frac{1}{2N} d_{axy}(C_x F_{yd} + F_{yd} C_x) - (\frac{1}{3} + \frac{1}{2N}) F_{ad} \\
&= \frac{1}{2N} d_{axy} \Big(i(-C_x C_d C_y + C_y C_d C_x + C_x C_y C_d - C_d C_y C_x) \\
&\quad + \frac{1}{2}(f_{ydz} C_x C_z + f_{ydz} C_z C_x) \Big) - (\frac{1}{3} + \frac{1}{2N}) F_{ad} \\
&= \frac{1}{2N} \Big([i(d_{axy} C_x C_y), C_d] + \frac{1}{2}(f_{dce} d_{eab} + f_{dbe} d_{eac}) C_b C_c \Big) - (\frac{1}{3} + \frac{1}{2N}) F_{ad} \\
&= \frac{1}{2N} \Big(i[(d_{axy} C_x C_y), C_d] + \frac{1}{2} f_{ade}(d_{ebc} C_b C_c) \Big) - (\frac{1}{3} + \frac{1}{2N}) F_{ad} \\
&= \frac{1}{2N}(i[D_a, C_d] + \frac{1}{2} f_{ade} D_e) = o(\mathcal{D}^{nl}(C)) + o([C_a, \mathcal{D}^{nl}(C)_b]) \qquad (112)
\end{aligned}
$$

using (29). The subscript $\mathcal{D}_1^{lin}$ indicates that $\mathcal{D}^{lin}$ is applied to the first index. Hence F_{ab} is indeed tangential with the above assumptions. Furthermore, note that the last term $f_{abc} A_c$ in the definition (98) of F_{ab} does not contribute for tangential A_a. Then the usual formula for F_{ab} is reproduced e.g. at the north pole.

5.3   The Yang-Mills action

Assume that the C_a satisfy the constraints (103) (and therefore also (104)) of $\mathbb{C}P_N^2$ exactly or approximately, so that F_{ab} is tangential as shown above. Then the "Yang-Mills" action is defined as

$$
S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g D_N} Tr(-[C_a, C_b]^2 + 2i f_{abc} C_a C_b C_c + 3 C_a C_a). \qquad (113)
$$

It reduces to the classical Yang-Mills action on $\mathbb{C}P^2$, because only the tangential indices contribute as shown in (112). S_{YM} can also be written in the form (87). The corresponding equation of motion is $-2[C_b, [C_a, C_b]] + 3i f_{abc} C_b C_c + 3 C_a = 0$, which is

$$
2[C_b, F_{ab}] - i F_a = 0 \qquad (114)
$$

The last term may seem strange, but it does not contribute to the tangential fields since $F_{\mathfrak{t}} \to 0$ as $N \to \infty$ by (112). Noting that $[C_b, .] = [\xi_b + A_b, .]$ corresponds to the covariant derivative in the adjoint, this is equivalent to the usual equation $D^\mu F_{\nu\mu} = 0$ for the tangential fields.

We now have to impose the constraints (103), (104) either exactly or approximately, and there are several possibilities how to proceed. Imposing both of them exactly seems too restrictive; recall that they are not independent even classically. One can hence either

1. consider all 8 fields C_a as dynamical and add

$$S_D = \frac{1}{gD_N} Tr\left(\mu_1(dCC - (\frac{2N}{3} + 1)C)^2 + \mu_2(C \cdot C - (\frac{N^2}{3} + N))^2\right) \quad (115)$$

to the action. This will impose the constraint dynamically for suitable $\mu_1 > 0$ and $\mu_2 \geq 0$, by giving the 4 transversal fields a large mass $m \to \infty$. Or,

2. impose the constraint $D = dCC - (\frac{2N}{3} + 1)C = 0$ exactly, or a slightly modified version.

In the second approach, it is not clear whether there are sufficiently many solutions of $D = 0$ in the noncommutative case to admit 4 tangential gauge fields. This concern could be circumvented by modifying the constraint, which is discussed in Section 5.5. However we have not been able to find instanton-like solutions in this case (which may just be a technical problem). Therefore we concentrate on the first approach here, where we do find topologically nontrivial instanton solutions. It offers the additional possibility to give physical meaning to the non-tangential degrees of freedom.

Therefore our action is

$$\boxed{S = S_{YM} + S_D.} \quad (116)$$

We will show that this reproduces the classical Yang-Mills action on $\mathbb{C}P^2$ in the large N limit for

$$\boxed{\mu_1 = o(\tfrac{1}{N}), \quad \text{and} \quad \mu_2 \leq o(\tfrac{1}{N^3}).} \quad (117)$$

Here $o(\frac{1}{N})$ stands for a function which scales like $\frac{1}{N}$. We proceed to find the "vacuum", i.e. the minimum of the action. Assume first that the size of the matrices is D_N; this will be somewhat relaxed below. Then the absolute minima of the action are given by solutions of $F_{ab} = 0$ and $D_a = 0$, which means that C_a is a representation of $su(3)$ with $D_a = 0$. The latter implies (e.g. using (222)) that the only allowed irreps are $V_{N\Lambda_2}$ or the trivial representation, and the trivial one is suppressed by μ_2. To determine the appropriate range for $\mu_{1,2}$, note that $\mu_1 = o(1/N)$ is sufficient by (132) to decouple the 4 transversal degrees of freedom. It implies that $\mathcal{D}^{nl}(C) \to 0$ for $N \to \infty$ (for finite action) as required in Section 5.2.1, hence it allows 4 tangential gauge fields. On the other hand, the instanton solutions found in Section 6 have $D_a = o(1)$ and also $C \cdot C - (\frac{N^2}{3} + N) = o(1)$, therefore we need $\mu_{1,2} \to 0$ to make sure S_D does not contribute to the Yang-Mills action in the commutative limit $N \to \infty$. Finally, we need $\mu_2 \leq o(1/N^3)$, otherwise the trivial blocks in (162) would be suppressed. All this leads to (117). One might allow $\mu_2 = 0$ and argue that the nontrivial vacuum is chosen due to the larger phase-space.

Therefore the "vacuum" solution is

$$(C_{vac})_a = \xi_a \quad (118)$$

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in a suitable basis, and other saddle points have a larger action. These arguments go through if we allow the size of the matrices C_a to be somewhat bigger that D_N, say

$$C_a \in Mat(D_N + N, \mathbb{C}) \tag{119}$$

(anything much smaller that $2D_N$ will do), which is needed to accomodate all the non-trivial solutions found below. Any configuration with finite action is therefore close to (118), and can hence be written as

$$C_a = \xi_a + A_a \tag{120}$$

in a suitable basis, with "small" A_a. This justifies the assumptions made in the beginning of Section 5.3.

Coupling to matter fields. To further clarify the physical meaning of this matrix-model approach to gauge theory, assume that we have an additional complex scalar field ϕ. Without gauge coupling, a natural action would be $\int ([\xi_a, \phi])^\dagger [\xi_a, \phi] = -\int \phi^\dagger \Delta \phi$. If we assume that ϕ is charged and transforms under gauge transformations as

$$\phi \rightarrow U\phi, \tag{121}$$

then a natural gauge-invariant action would be

$$S[\phi] = \int (C_a \phi - \phi \xi_a)^\dagger (C_a \phi - \phi \xi_a). \tag{122}$$

This reduces to $\int (D_a \phi)^\dagger D_a \phi$ where $D_a = [\xi_a, .] + A_a$. The combined action $S_{YM} + S[\phi]$ is again a matrix model, and we should expect solutions where both C_a and ϕ are block-matrices in $End(V)$. In particular, ϕ may effectively be a rectangular matrix $Hom(V, V')$ (the rest being filled by zeros, say). This is exactly what happens for the monopole and instanton solutions constructed below, where $C_a \in End(V') \subset End(V)$ couples naturally to $\phi \in Hom(V, V')$. The latter should therefore be considered as a section in an associated (nontrivial) bundle, which will be made very explicit in Section 6.2.3. This is in complete agreement with the results of [22, 27], establishing an indirect connection with the approach using projective modules.

The appropriate action for fermions is not obvious since $\mathbb{C}P^2$ has no spin but spin^c structure. Some proposals for a Dirac operator have been given in the literature [14, 15], and we postpone the formulation in our approach to future work.

Rewriting the action and equations of motion. One important observation is that using some identities of the $su(3)$ tensors (see (228), Appendix C), one can rewrite the YM term as

$$S_{YM} = \frac{1}{gD_N} Tr \left(2(C \cdot C)(C \cdot C) - \frac{3}{2}(dCC)(dCC) - \frac{1}{2}(fCC)(fCC) + 2ifCCC + 3C \cdot C \right) \tag{123}$$

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where the obvious index structure has been omitted, and $C \cdot C = C_a C_a$. Using this and

$$
\begin{aligned}
Tr((\delta(fCC))X_a) &= Tr(\delta C_a(f_{abc}[C_b, X_c])), \\
Tr((\delta(dCC))X_a) &= Tr(\delta C_a(d_{abc}\{C_b, X_c\}))
\end{aligned}
\tag{124}
$$

the equations of motion for S_{YM} can be written as

$$
4\{C_a, (C \cdot C)\} + 6C_a - 3d_{abc}\{C_b, (dCC)_c\} + if_{abc}[C_b, F_c] = 0.
\tag{125}
$$

This is much easier to work with. S_D leads to the additional term

$$
\mu_1\Big(2d_{abc}\{C_b, (dCC)_c\} - 6(2N/3+1)d_{abc}C_bC_c + 2(2N/3+1)^2C_a\Big) + 2\mu_2\{C_a, C\cdot C - (\frac{N^2}{3}+N)\}
\tag{126}
$$

in the lhs of (125). A particularly interesting form (156) is obtained for $\mu_1 = 2, \mu_2 = -\frac{2}{3}$, which is however outside of the range (117) where the model is under control.

5.4 Decoupling of auxiliary variables

As discussed above, we impose the constraints of $\mathbb{C}P_N^2$ by adding the term

$$
S_D = \frac{1}{gD_N} Tr\left(\mu_1 D_a D_a + \mu_2(C \cdot C - (\frac{N^2}{3} + N))^2\right)
\tag{127}
$$

to the action. We will now show that this amounts to giving the 4 transversal fields a large mass $m \to \infty$ which therefore decouple, leaving 4 massless tangential gauge fields. In fact one can put $\mu_2 = 0$, since (104) is not an independent constraint. Using $C_a = \xi_a + A_a$ we get

$$
\begin{aligned}
D_c &= d^{ab}{}_c\{\xi_a, A_b\} + d^{ab}{}_c A_a A_b - (\frac{2N}{3} + 1)A_c \\
C_a C^a - (\frac{N^2}{3} + N) &= \xi_a A^a + A_a \xi^a + A_a A^a.
\end{aligned}
\tag{128}
$$

In particular, assuming that A_a and $[A, A]$ are "smooth" we have

$$
\frac{D_c}{2N} = \frac{1}{2N}d^{ab}{}_c\{\xi_a, A_b\} - \frac{1}{3} A_c + o(1/N) = \mathcal{D}^{lin}(A_c) + o(1/N)
\tag{129}
$$

If A_a are (the quantization (56) of) regular classical tangential fields, we can assume that $\mathcal{D}^{lin}(A_c) = o(1/N)$ using results in [14]. Therefore $\frac{D_c}{N} \approx 0$ for such tangential fields. On the other hand, $\mathcal{D}^{lin}(A)$ reproduces the transversal fields with their respective eigenvalue -1 resp. $\frac{1}{3}$. To see this explicitly, consider the "north pole", where $x_a \approx \delta_{a,8}$. Then

$A_t = A_{4,5,6,7}$ are tangential, and $A_{1,2,3}$ and A_8 are "transversal" with

$$\begin{aligned}
\frac{D_t}{2N} &= o(1/N), \\
\frac{D_{1,2,3}}{2N} &= -A_{1,2,3} + o(1/N), \\
\frac{D_8}{2N} &= \frac{1}{3}A_8 + o(1/N), \\
\frac{1}{2N}(C_a C^a - (\frac{N^2}{3} + N)) &= \frac{1}{\sqrt{3}}A_8 + o(1/N). \tag{130}
\end{aligned}$$

This shows how D_a separates the tangential from the transversal fields. Therefore the term $\mu_1 D_a D_a$ gives the transversal modes[8] A_t a mass term of order $\mu_1 N^2$, while the tangential modes are affected by terms of order at most μ_1. Similar conclusions can be drawn for μ_2. This means that if we choose

$$\mu_1 = 1/N, \tag{131}$$

then

$$\mu_1 D_a D_a = 4N(A_1^2 + A_2^2 + A_3^1 + \frac{1}{9}A_8^2) \tag{132}$$

(at the north pole), and similar for the other term. This implies that our action (116) indeed approaches the classical Yang-Mills action for $N \to \infty$.

Note that for the special choice $\mu_1 = 2, \mu_2 = -\frac{2}{3}$ considered in Section 5.7, the mass term for A_8 is undetermined. The meaning of the model is then unclear.

Additional terms in the action. Based on $su(3)$ invariance, one should also allow other terms such as

$$\int a_1 \, C \cdot C + a_2 \, fCCC + a_3 \, dCCC \tag{133}$$

etc in the action. However, recall that such $su(3)$ singlets may *not* be invariant under local $so(4)$ rotations in the commutative limit. Consider first the terms $dCCC$ and $C \cdot C$. They are clearly related to the constraints (103) and (104), hence they are essentially constants, and therefore harmless for small enough coefficients a_i. They are also covariant in the usual sense, which can be seen using the explicit form of the d_{abc}. The term $fCCC$ is less obvious at first sight, since it is not covariant in the usual sense. This may seem very dangerous: One might of course exclude this term from the bare action, however quantum corrections are likely to restore it. Fortunately, it is again harmless: according to (92), it essentially reduces to the first Chern number (plus a constraint) in the classical limit, which is topological and does not affect the local physics as long as $a_2 = o(1/N)$. Using (91), we see that the explicit breaking of covariance is due to the symplectic form η which lives on $\mathbb{C}P^2$.

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[8]the splitting into tangential and transversal modes is thereby defined in a global way

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We note in passing that in string theory or boundary CFT [19], fuzzy spaces arise as D-branes on group manifolds, and a term of the form $fCCC$ occurs in the effective action (the "Chern-Simons" term). We now understand that this is a combination of the first Chern number and a Casimir-constraint. This interpretation is expected to hold also for higher-dimensional branes. However, the constraints (103), (103) do not arise in this context, hence the action in [19] has a completely different physical meaning.

5.5 Alternative approach: constraints

As discussed above, imposing both (103), (104) in the fuzzy case is probably too restrictive. The first remedy is to drop (104), because it is redundant classically and is not compatible with the monopole solutions below. Hence consider $D_a = 0$ only. However, it appears that even this is too strong, at least we see no reason why these 8 equations should admit 4 independent degrees of freedom; note that the kernel of $\delta \mathcal{D}^{nl} = \mathcal{D}^{lin}$ is probably smaller than classically. To find a way out, note first that a slightly modified constraint

$$D_a = d_{abc} C_b C_c - (\frac{2N}{3} + 1)C_a = -F_a \tag{134}$$

would also be acceptable, because it implies $\mathcal{D}^{lin}(A) = o(1/N)$ as long as F is finite; this is all we needed above. This suggests to consider the following slightly generalized constraint, which indeed admits 4 tangential degrees of freedom: introducing a further (auxiliary) scalar field C_0, consider

$$C = C_0 + C_a \tau^a, \tag{135}$$

and impose the constraint

$$C^2 = \frac{(N+1)^2}{4}. \tag{136}$$

This means that C has 2 eigenvalues $\pm \frac{N+1}{2}$, and the multiplicities $n_\pm$ are determined by $Tr(C)$. In component form, this is equivalent to

$$\frac{2}{3} C_a C_a + C_0^2 - \frac{(N+1)^2}{4} = 0 = (if_{abc} + d_{abc})C_a C_b + 4\{C_0, C_c\}. \tag{137}$$

If $C_0 \approx const$ (which could be enforced by adding a term $[C_a, C_0]^2$ to the action), then the first equation implies $C_0 \approx \frac{-N+3}{6}$, and the second equation reduces to (134). In particular, this is solved by

$$C = \frac{-N+3}{6} + \xi_a \tau_a. \tag{138}$$

Note that C_0 is essentially determined by the first equation, revealing its auxiliary nature.

It is easy to see that (136) admits 4 tangential degrees of freedom: Indeed, the most general variation of some C consistent with this constraint has the form

$$\delta C = [X, C] \approx iN \mathcal{J}(X) \tag{139}$$

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for an arbitrary matrix $X = X_a \tau_a$, cp. (107). Since $\mathcal{J}$ projects out the radial degrees of freedom, only the tangential fields survive and can be chosen freely.

A further advantage of this approach is the fact that the path integral is an integral over the compact space $U(3D)/(U(n_+) \times U(n_-))$. Also, it turns out that the monopole and instanton configurations below are compatible with that constraint[9]. All this is quite appealing. However, these instantons turn out to be no longer solutions of the corresponding equation of motion. The significance of this fact is not clear; it might be that there are other "nearby" solutions in this case, or it might indicate some different global implications of this constraint. Furthermore, the addition of the field C_0 somewhat complicates the analysis. We therefore concentrate on the approach with auxiliary variables.

5.6 Nonabelian case: $U(n)$ Yang-Mills

The generalization to a $U(n)$ gauge theory is straightforward as in [11], and is achieved by considering the same action for larger matrices C_a. In order to accomodate the monopole and instanton solutions constructed below, we allow the size to be somewhat larger that nD_N, say

$$C_a \in Mat(n(D_N + N), \mathbb{C}) \tag{140}$$

(anything much smaller that $(n+1)D_N$ will do). Then consider the same matrix model (116) as before.

The following analysis is parallel to that in Section 5.3. First we should find the ground state. For $\mu_2 = 0$, the absolute minima of the action are given by solutions of $D_a = 0$ and $F_{ab} = 0$, which means that C_a is a representation of $su(3)$ with $D_a = 0$. The latter implies that the representation decomposes into a direct sum of either $V_{N\Lambda_2}$ or the trivial rep. The trivial representations are suppressed by making μ_2 slightly bigger than zero, and in a suitable basis C takes the form

$$(C_{vac})_a = \xi_a \, \mathbf{1}_{n \times n} \tag{141}$$

which is a block matrix consisting of n blocks of the fuzzy $\mathbb{C}P^2$ solutions. The action is then zero, and clearly all other saddle points have a positive action. Any configuration with finite action is therefore close to (141), and can be written as

$$C_a = \xi_a \, + A_a \tag{142}$$

where A_a is "small" and carries an additional $u(n)$ index,

$$A_a = A_{a,\alpha} \, \lambda_\alpha. \tag{143}$$

Here λ_α denote the Gell-Mann matrices of $u(n)$, and $\lambda_0 = \mathbf{1}$ is the $n \times n$ unit matrix. The rest of the analysis of the previous sections goes essentially through, in particular the

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[9]this motivates introducing $\tilde{D}$ in (199), because it satisfies a quadratic characteristic equation

transversal components of $A_{a,\alpha}$ will be very massive and decouple due to S_D. The field strength is again $F = dA + AA$ or (98) in component notation, which becomes the usual field strength of a $U(n)$ gauge theory if the $u(n)$ indices are spelled out. In particular, all non-tangential components are suppressed again due to (112), and the equations of motion have the same form (114) or (125), (126). All this will be understood in the following.

5.7 One- and Two-Matrix Models for Fuzzy $\mathbb{C}P_N^2$

It is tempting to consider also the analog of the single-matrix model studied in [11] for the fuzzy sphere. Consider a hermitian $3D_N \times 3D_N$ matrix

$$C = C_0 \mathbb{1} + C_a \tau_a = \frac{-N+3}{6} \mathbb{1} + C_a \tau_a. \tag{144}$$

Then

$$C^2 = \left(\frac{2}{3} C_a C_a + C_0^2\right) \mathbb{1} + \left(C_a C_b \frac{1}{2}(i f_{abc} + d_{abc}) + (C_0 C_c + C_c C_0)\right) \tau_c. \tag{145}$$

If we set $C_0 = \frac{-N+3}{6}$, we have

$$C^2 - \frac{(N+1)^2}{4} = \frac{2}{3}\left(C_a C_a - N(\frac{N}{3}+1)\right)\mathbb{1} + \frac{1}{2}(F_c + D_c)\,\tau_c \tag{146}$$

for F_a, D_a as defined before, hence $C_{vac}^2 = \frac{(N+1)^2}{4}$ for

$$C_{vac} = \frac{-N+3}{6} + \xi_a \tau_a. \tag{147}$$

Now consider the action [11]

$$
\begin{aligned}
S &= \frac{1}{gD_N} Tr V(C) = \frac{1}{gD_N} Tr\left((C^2 - \frac{(N+1)^2}{4})^2\right) \\
&= \frac{1}{g}\int \frac{4}{3}\,(C_a C^a - (\frac{N^2}{3}+N))^2 + \frac{1}{2}(F_c + D_c)(F_c + D_c).
\end{aligned} \tag{148}
$$

As opposed to the 2-dimensional case, this does not seem to describe an interesting gauge theory on fuzzy $\mathbb{C}P^2$, because the auxiliary fields "eat up" the gauge fields: Since for a C with finite action the eigenvalues and multiplicities of C must be approximately the same as those of C_{vac}, there exists a basis where $C_a = \xi_a + A_a$ for "small" A_a. We can then split A_a into tangential and transversal components as before, and at the north pole this action is

$$S = \frac{1}{g}\int \frac{16}{9}\,N^2 A_8 A_8 + \frac{1}{2}\Big(\sum_{r=1,2,3}(-2N A_r + F_r)^2 + (\frac{2N}{3} A_8 + F_8)^2 + \sum_{t=4}^{7}(D_t + F_t)^2\Big) \tag{149}$$

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(at the north pole). Since the A_a are small we can assume that F_a is finite, so that $A_{\mathfrak{r}} = o(\frac{1}{N})$ and $F_{\mathfrak{t}} = o(\frac{1}{N})$. Hence A is indeed tangential, and after integrating out the $A_{\mathfrak{r}}$ one is left with

$$S = \frac{1}{g} \int \frac{1}{2} \sum_{t=4}^{7} (D_t + F_t)^2. \tag{150}$$

The meaning of this is obscure.

Conjugate fields The situation becomes somewhat more interesting if one adds a second "conjugated" field

$$\tilde{C} = \frac{N+6}{6} \, \mathbb{1} \, + C_a \tilde{\tau}_a \tag{151}$$

where ˜denotes the outer (diagram) automorphism of $su(3)$, i.e. essentially $\tilde{\tau}_a = \lambda_a$ are the ordinary Gell-Mann matrices. Alternatively, the complex conjugate τ_a^* are also equivalent to $\tilde{\tau}_a$. Then

$$\tilde{\tau}_a \tilde{\tau}_b = \frac{2}{3} \delta_{ab} + \frac{1}{2} (i f_{abc} - d_{abc}) \tilde{\tau}_c, \tag{152}$$

the only difference to (5) being the sign in front of d_{abc}. Then

$$\tilde{C}^2 - \frac{(N+2)^2}{4} = \frac{2}{3} \left(C_a C_a - N(\frac{N}{3}+1) \right) \mathbf{1} + \frac{1}{2} (F_c - D_c) \, \tau_c \tag{153}$$

for F_a, D_a as defined before. Therefore $\tilde{C}^2 = \frac{(N+2)^2}{4}$ for $C_a = \xi_a$, and we can consider the action

$$\begin{aligned}
\tilde{S} &= \frac{1}{g D_N} Tr \tilde{V}(\tilde{C}) := \frac{1}{g D_N} Tr \left((\tilde{C}^2 - \frac{(N+2)^2}{4})^2 \right) \\
&= \frac{1}{g} \int \frac{4}{3} \, (C_a C_a - (\frac{N^2}{3}+N))^2 + \frac{1}{2} (F_c - D_c)(F_c - D_c). \tag{154}
\end{aligned}$$

Therefore

$$S + \tilde{S} = \frac{1}{g D_N} Tr(V(C) + \tilde{V}(\tilde{C})) = \frac{1}{g} \int \frac{8}{3} \, (C_a C_a - (\frac{N^2}{3}+N))^2 + F_a F_a + D_a D_a. \tag{155}$$

Now the transversal $A_{\mathfrak{r}}$ (at the north pole) decouple due to $D_a D_a$, and one is left with essentially $Tr(F_{\mathfrak{r}} F_{\mathfrak{r}} + D_{\mathfrak{t}} D_{\mathfrak{t}})$. However, this is still not what we want because $F_a F_a$ is not covariant in the usual (Euclidean) sense, and furthermore the meaning of the tangential $D_{\mathfrak{t}} D_{\mathfrak{t}}$ is not clear. This is related to the question whether one can solve the constraint $D_a = 0$ exactly in the noncommutative case, or only up to $o(1)$ corrections.

In particular, we make the following observation: for the special choice $\mu = 2$ and $\mu_2 = -\frac{2}{3}$, the gauge action (116) can be written as

$$S_{YM} + S_D = \frac{1}{g D_N} Tr(\frac{4}{3}(C \cdot C)(C \cdot C) + \frac{1}{2}(dCC)(dCC) + \frac{1}{2}(ifCC)(ifCC)$$

$$+2if_{abc}C_aC_bC_c - 4(\frac{2N}{3}+1)dCCC + (\frac{4N^2}{3}+4N+5)C_aC_a)$$
$$= \frac{1}{gD_N} Tr(V_1(C) + V_2(\tilde{C})) \tag{156}$$

for

$$
\begin{aligned}
V_1(C) &= \frac{1}{2}(C + \frac{N-3}{6})^2(C^2 - \frac{1}{3}(7N+9)C + \frac{1}{12}(N+3)(11N+25)), \\
V_2(\tilde{C}) &= \frac{1}{2}(\tilde{C} - \frac{N+6}{6})^3(\tilde{C} + \frac{5}{2}(N+2))
\end{aligned}
\tag{157}
$$

While the meaning of this model is unclear for the same reasons as above and furthermore the mass term for A_8 is undetermined (see Section 5.4), the above form might be accessible to analytical (or numerical) studies. This certainly motivates further investigations.

5.8 Quantization

The quantization of these models is straightforward in principle, by a "path integral" over the hermitian matrices

$$Z[J] = \int dC_a e^{-(S_{YM}+S_D+TrC_aJ_a)} \tag{158}$$

Note that there is no need to fix the gauge unless one wants to do perturbation theory, since the gauge orbit is compact. The gauge-fixing terms required for perturbation theory will not be discussed here, cp. [12].

We claim that the above path integral is well-defined and finite for any fixed N provided $\mu_1 > 0$ and $\mu_2 \geq 0$. To see this, note e.g. by rescaling $C_a \to \alpha C_a$ that it is enough to show that $\int dC_a e^{-Tr((i[C,C])^2+\mu_1(dCC)^2)}$ is convergent, in obvious notation. Using $Tr(i[C,C])^2 \geq Tr\frac{1}{\sqrt{12}}(ifCC)^2$ and (228), it is enough to show that $\int dC_a e^{-Tr(C\cdot C)^2}$ is convergent, which is true.

For the approach with constraints in Section 5.5, the path integral is over the constrained configuration space $C^2 = \frac{(N+1)^2}{2}$, which is an integral over the compact space $U(3D_N)/(U(n_+) \times U(n_-))$. In either case, this finiteness property allows to study quantum field theory on $\mathbb{C}P_N^2$ in a very clean way. In both approaches, the nontrivial question of course remains whether the model is renormalizable, i.e. whether there exists a suitable scaling of the coefficients in the action such that the limit $N \to \infty$ of the quantized model is well-defined. These issues could be studied using the renormalization group methods developed in [28].

6 Topologically nontrivial solutions

In this section, we will construct some explicit solutions of the equations of motion (125), (126) with finite action, which in the classical (large N) limit become topologically nontrivial solutions such as monopoles and instantons.

The idea is to identify the solutions of the gauge theory with certain irreps of the symmetry group $SU(3)$, which replaces the Poincare group. Recalling that the "vacuum" solution $C_a = \xi_a = \pi_{(0,N)}(T_a)$ of our action $S_{YM} + S_D$ is obtained as irrep $V_{(0,N)}$, it is natural to consider other representations, such as $C_a = \pi_\Lambda(T_a)$ for other irreps labeled by their highest weight Λ. If Λ is close to $(0, N)$, it seems plausible that they give rise to nontrivial saddle-points of the action. This idea essentially works, with some modifications which are necessary in the nonabelian case.

6.1 Monopoles, or $U(1)$ instantons

In the classical case, $\mathbb{C}P^2$ admits monopole configurations with any integer first Chern number or charge, due to the presence of a nontrivial 2-sphere which generates $H_2(\mathbb{C}P^2)$. They have been constructed on fuzzy $\mathbb{C}P^2$ as projective modules in [22]. Here we will recover them as solutions of the equation of motion.

Consider the Ansatz

$$C_a = \alpha \xi_a^{(M)} = \alpha \pi_{(0,M)}(T_a) \tag{159}$$

where $\xi_a^{(M)}$ is the generator of $su(3)$ in the representation $V_{M\Lambda_2}$, with

$$M = N - m. \tag{160}$$

The gauge field $A_a(x)$ is then determined by considering C_a as a fluctuation of the vacuum $\xi_a = \xi_a^{(N)}$, i.e.

$$C_a = \xi_a + A_a. \tag{161}$$

This means that one should imagine C_a to be a $D_M \times D_M$ block-matrix embedded in the configuration space of $D_N \times D_N$ matrices as in [11]:

$$C_a = \begin{pmatrix} \alpha \xi_a^{(M)} & 0 \\ 0 & 0 \end{pmatrix} = \xi_a + A_i \sigma^i \tag{162}$$

(note that $C_a = 0$ is also a solution of the equation of motion with action 0, provided $\mu_1 = 0$). This is clear as long as $M < N$. However we also want to admit $M > N$, which is achieved by relaxing the condition that C_a be $D_N \times D_N$ matrices. For example, one could fix C_a to have size $D_N + N$, which for large N admits all relevant solutions of an abelian gauge theory. Notice also that the particular embedding above determines the

location of a "Dirac string" (singularity of the gauge field). This embedding is of course arbitrary, and the Dirac string can be moved using a gauge transformation.

The precise normalization α is determined by the equation of motion and depends on μ, but essentially it is determined by the constraint $D_a = o(1)$. Using

$$[\xi_a^{(M)}, \xi_b^{(M)}] = \frac{i}{2} f_c^{ab} \xi_c^{(M)}, \qquad d_c^{ab} \xi_a^{(M)} \xi_b^{(M)} = (\frac{2M}{3} + 1) \, \xi_c^{(M)} \tag{163}$$

one finds that $D_a = o(1)$ implies

$$\alpha = 1 + \frac{m}{N} + o(1/N^2) \tag{164}$$

where the $o(1/N^2)$ term depends on μ. This also implies $C_a C_a = \frac{1}{3} N^2 + N + o(1)$, hence (159) describes indeed a tangential gauge field on fuzzy $\mathbb{C}P^2$. The gauge field is then guaranteed to be tangential (112), and is given by

$$F_{ab} = \frac{1}{2} \, f_{ab}^c (-\alpha^2 + \alpha) \xi_c^{(M)} \approx -\frac{m}{2} \, f_{ab}^c \frac{\xi_c^{(M)}}{N}. \tag{165}$$

A_a turns out to be finite except at the "south sphere", and will be calculated explicitly below. This implies that

$$\frac{\xi_a^{(M)}}{N} = \frac{\xi_a}{N} + o(1/N) \approx \frac{x_a}{\sqrt{3}}, \tag{166}$$

hence

$$F_{ab} = -\frac{m}{2\sqrt{3}} \, f_{ab}^c x_c, \tag{167}$$

or

$$= -2i\pi m \, \omega \tag{168}$$

is a multiple of the symplectic form $\eta = 3\pi\omega$ (88). In particular, F is selfdual. Using the formulas in Section 4.1, the first Chern number in the large N limit is

$$c_1 = m. \tag{169}$$

This shows that A_a is indeed a connection on a bundle, and we have recovered the result of [22] in a somewhat different formulation. Such monopoles do not exist on S^4. The value of the action is

$$S_{YM} = \frac{1}{g} \int F_{ab} F_{ab} = \frac{1}{g} \int 12m^2 \tag{170}$$

plus corrections of order $o(1/N)$, and S_D does not contribute for large N.

6.1.1 Explicit form of the gauge field

We want to calculate the gauge field

$$A_a^{mono}(x) = \alpha \xi_a^{(M)} - \xi_a \tag{171}$$

explicitly. We fix the gauge (using a $U(D_M)$ gauge transformation) by diagonalizing both $\xi_3^{(M)}$ and $\xi_8^{(M)}$, and matching the eigenvalues with those of $\xi_{3,8}$ except for the lowest eigenvalues of ξ_8. This amounts to a embedding of the corresponding matrices, and putting the singularity at the south sphere. Geometrically, it amounts to match the "upper parts" of the weight triangles of the representations $V_{M\Lambda_2}$ and $V_{N\Lambda_2}$. Since all multiplicities are one, one could simply work with the $su(2)$ subalgebras. However as a warm-up for the instanton calculation, we shall use the Gelfand-Tsetlin basis, where the operators can be calculated explicitly. One finds that (see (254) in Appendix F) the operators $\xi_a^{(M)}$ can be written in terms of the ξ as

$$
\xi_{1,2,3}^{(M)} = \xi_{1,2,3}, \qquad \xi_8^{(M)} = \xi_8 - \frac{m}{\sqrt{3}},
$$

$$
\xi_4^{(M)} \pm i\xi_5^{(M)} = (\xi_4 \pm i\xi_5) - m\frac{\sqrt{3}}{2}\frac{x_4 \pm ix_5}{2x_8 + 1},
$$

$$
\xi_6^{(M)} \pm i\xi_7^{(M)} = (\xi_6 \pm i\xi_7) - m\frac{\sqrt{3}}{2}\frac{x_6 \pm ix_7}{2x_8 + 1} \tag{172}
$$

using $x_a = \frac{\sqrt{3}}{N}\xi_a$. There are further correction terms of order $\frac{1}{N(2x_8+1)^2}x$, which vanish for large N as long as $2x_8 + 1 > 0$, i.e. away from the south sphere. We can now find α_m by requiring that $A_8 = 0$ at the north pole, which using $\xi_8|_{NP} = \frac{N}{\sqrt{3}}$ gives

$$\alpha = \frac{N}{M} = 1 + \frac{m}{N} + o(1/N^2) \tag{173}$$

in agreement with (164). Then the gauge field for large N is

$$
A_{1,2,3}^{mono} = \frac{m}{\sqrt{3}} x_{1,2,3}, \qquad A_8^{mono} = \frac{m}{\sqrt{3}}(x_8 - 1),
$$

$$
A_4^{mono} \pm iA_5^{mono} = \frac{m}{\sqrt{3}}(1 - \frac{3}{2}\frac{1}{2x_8 + 1})(x_4 \pm ix_5),
$$

$$
A_6^{mono} \pm iA_7^{mono} = \frac{m}{\sqrt{3}}(1 - \frac{3}{2}\frac{1}{2x_8 + 1})(x_6 \pm ix_7).
$$

In the classical limit, these formulas are valid everywhere except at the lowest eigenvalue of $2x_8 = -1$, i.e. the south sphere. One can check easily that $x_a A_a = 0$.

6.2 Nonabelian case: $U(2)$ instantons

We will exhibit here nontrivial solutions of the equation of motion (125) for the nonabelian $U(2)$ gauge theory defined by the action (116), for matrices C_a of size $\approx 2D_N$. In the

classical limit, they describe $U(2)$ gauge fields with nontrivial first and second Chern number and finite action.

We want to generalize the above construction for other representations such as $V_{(1,N)}$. Here some modification is required, which can be understood as follows: according to [24], $C'_a = \pi_{(1,N)}(T_a)$ can be considered as quantization of a 6-dimensional adjoint orbit infinitesimally close to $\mathbb{C}P_N^2$, which can (heuristically) be viewed as "bundle" over $\mathbb{C}P_N^2$ whose fiber is a non-commutative 2-point space. This C'_a does not satisfy the constraint $D_a = 0$ of $\mathbb{C}P^2$, and we have to modify C'_a; hence the prime. This can be done using the map $\mathcal{D}^{nl}$ defined in Section 5.2, which leads to an instanton-like solution. There is a small caveat: our construction will only give us the uniform, non-localized instanton with instanton number 1, and not the full moduli space of instantons [29]. One may hope that a modification of the construction presented here will also give localized instantons. Furthermore, our solutions are not (anti)selfdual since they also contain certain $U(1)$ monopoles.

Even though we do not attempt it here, it seems plausible that the generalization of this construction to $V_{(k,N)}$ leads to solutions with instanton number k.

6.2.1 Group-theoretical origin

Before exhibiting an exact solution of our model, we give an approximate derivation which is valid for large N. From a group-theoretical point of view, there is a natural candidate for a nontrivial saddle point generalizing (159) given by the Ansatz

$$C''_a = \zeta_a^{(m)} = \pi_{M\Lambda_2 + \Lambda_1}(T_a) \tag{174}$$

where $\zeta_a^{(m)}$ is the representation with highest weight $M\Lambda_2 + \Lambda_1$, and

$$M = N - m.$$

Since $\dim(V_{N\Lambda_2 + \Lambda_1}) \approx 2\ \dim(V_{N\Lambda_2})$, we can hope to embed $V_{M\Lambda_2+\Lambda_1}$ in $V_{N\Lambda_2} \otimes \mathbb{C}^2$, defining a gauge field by

$$A''_a = \zeta_a^{(m)} - \xi_a. \tag{175}$$

In fact, we will see below that A''_a is finite and well-defined in the classical limit (except at the south sphere). Unfortunately, the above Ansatz is not admissible because $A''_{\mathfrak{t}}$ is not tangential. However, this can be fixed in a $su(3)$-covariant and gauge invariant way, by subtracting the transversal components using the tensor maps $\mathcal{D}^{lin}$ or $\mathcal{D}^{nl}$. Recall that the linear map $\mathcal{D}^{lin}$ separates the tangential from the transversal components, and splits the latter into two subspaces: one component parallel to ξ_a (i.e. A''_8 at the north pole) with approximate eigenvalue $+\frac{1}{3}$, and the complement (i.e. $A''_{1,2,3}$ at the north pole) with approximate eigenvalue -1. The first can be absorbed simply by redefining

$$C'_a = \alpha' \zeta_a^{(m)} = \xi_a + A'_a$$

33

such that $C' \cdot C' = \xi \cdot \xi$ and hence $x \cdot A' = 0$. This leads to

$$\alpha' = 1 + \frac{2m - 1}{2N} + o(1/N^2). \tag{176}$$

Now the remaining non-tangential components have eigenvalue -1 of $\mathcal{D}^{lin}$, and hence can be subtracted by adding $\mathcal{D}^{lin}$. Since $\mathcal{D}^{lin}$ acting on C coincides with $\mathcal{D}^{nl}$, this leads to the modified Ansatz

$$\begin{aligned}
C_a &= C'_a + \frac{1}{2N}(d_{abc}C'_b C'_c - (\frac{2N}{3} + 1)C'_a) \\
&= \alpha\zeta_a^{(m)} + \beta(d_{abc}\zeta_b^{(m)}\zeta_c^{(m)} - (\frac{2M + 7}{3})\zeta_a^{(m)})) + o(1/N^2)\,\zeta_a^{(m)}
\end{aligned} \tag{177}$$

where the last equality holds for

$$\alpha = 1 + \frac{m}{N}, \qquad \beta = \frac{\alpha^2}{2N} = \frac{1}{2N} + \frac{2m - 1}{2N^2} + o(1/N^3). \tag{178}$$

Therefore the gauge fields A defined by

$$C_a = \xi_a + A_a \tag{179}$$

are finite, and satisfy $D^{lin}(A) = o(1/N)$. Hence the non-tangential components of A are of order $1/N$, or equivalently $dCC - (\frac{2N}{3} + 1)C = o(1)$ and $C \cdot C - \frac{N^2}{3} + N = o(1)$.

To summarize, we define the (approximately tangential) gauge fields by[10]

$$C_a = \alpha\zeta_a^{(m)} + \beta\tilde{D}_a^{(m)} = \xi_a + A_a \tag{180}$$

where

$$\tilde{D}_a^{(m)} = d_a^{bc}\zeta_b^{(m)}\zeta_c^{(m)} - \frac{1}{3}(2M + 7)\zeta_a^{(m)}. \tag{181}$$

In fact $\zeta_a^{(m)}$ and $\tilde{D}_a^{(m)}$ are the only two tensor operators on $V_{N\Lambda_2 + \Lambda_1}$ with an index in the adjoint. This strongly suggests that there should be an exact solution for the above Ansatz (180). This is indeed the case, and the result derived in Section 6.3 is in agreement with this approximate derivation. But first, we compute the corresponding field strength for large N.

6.2.2 Field strength, action and Chern class

Before showing that (180) indeed contains an exact solution of our model, we can get some insight by evaluating the field strength approximately at the north pole.

---

[10]the particular form of $\tilde{D}^{(m)}$ will be understood later

Since the gauge fields and field strength are operators on $V_{N\Lambda_2+\Lambda_1}$, there is a natural action of $su(3)$ on gauge fields, generated by the vector field resp. Lie derivative

$$\mathcal{L}_a = [\zeta_a, .] \tag{182}$$

It corresponds to a $SU(3)$ Killing vector field[11] in the classical limit. We extend its action to tensor fields with indices in $su(3)$ in the natural way (cp. [26]), in particular

$$\mathcal{L}_a C_b = [\zeta_a, C_b] - \frac{i}{2} f_{abc} C_c \tag{183}$$

and similarly for higher-rank tensor fields. It satisfies the $su(3)$ algebra

$$[\mathcal{L}_a, \mathcal{L}_b] = \frac{i}{2} f_{abc} \mathcal{L}_c. \tag{184}$$

In particular, the field strength transforms as

$$\mathcal{L}_a F_{bc} = [\zeta_a, F_{bc}] - \frac{i}{2}(f_{abd} F_{dc} + f_{acd} F_{bd}). \tag{185}$$

This can be used to rotate any "point" to the north pole, and reduce the computation of $F_{ab}(x)$ to a calculation it at the north pole. Now if C_a is the above instanton (180), covariance (201) implies that both the instanton field C_a as well as the corresponding field strength are *invariant* under this action of $SU(3)$. Therefore F_{ab} is constant apart from a rotation of the indices.

Let us therefore evaluate F_{ab} at the north pole. In view of the first line of (177), we have $C_a = \alpha' \zeta_a^{(m)} + \mathcal{D}^{lin}(A')_a + o(1/N) = \xi_a + A_a$. Therefore

$$C_{\mathfrak{t}} \approx \alpha' \zeta_{\mathfrak{t}}^{(m)}, \quad C_{\mathfrak{r}} \approx \xi_{\mathfrak{r}} \tag{186}$$

($\approx$ stands for equal up to $o(1/N)$) at the north pole, because the correction term $\mathcal{D}^{lin}(A')$ is transversal. This is also sufficient to calculate commutators at the north pole, as long as the fields are smooth for large N. Then using $\alpha' \zeta_{\mathfrak{r}}^{(m)} = \xi_{\mathfrak{r}} + A'_{\mathfrak{r}}$ we get

$$F_{\mathfrak{t}\mathfrak{t}} \approx \alpha'^2 i[\zeta_{\mathfrak{t}}^{(m)}, \zeta_{\mathfrak{t}}^{(m)}] + \frac{1}{2} f_{\mathfrak{t}\mathfrak{t}\mathfrak{r}} C_{\mathfrak{r}} \approx \frac{1}{2} f_{\mathfrak{t}\mathfrak{t}\mathfrak{r}}(-\alpha'^2 \zeta_{\mathfrak{r}}^{(m)} + \xi_{\mathfrak{r}}) = \frac{1}{2} f_{\mathfrak{t}\mathfrak{t}\mathfrak{r}}(\xi_{\mathfrak{r}}(1 - \alpha') - \alpha' A'_{\mathfrak{r}})$$

$$\approx \frac{1}{2} f_{\mathfrak{t}\mathfrak{t}\mathfrak{r}}(\frac{\frac{1}{2} - m}{\sqrt{3}} x_{\mathfrak{r}} - \alpha' A'_{\mathfrak{r}}) \tag{187}$$

$$F_{\mathfrak{t}\mathfrak{r}} \approx \alpha' i[\zeta_{\mathfrak{t}}^{(m)}, C_{\mathfrak{r}}] - \frac{1}{2} f_{\mathfrak{t}\mathfrak{r}\mathfrak{t}} C_{\mathfrak{t}} = \frac{1}{2} f_{\mathfrak{t}\mathfrak{r}\mathfrak{t}}(\alpha' C_{\mathfrak{t}} - C_{\mathfrak{t}}) = \frac{1}{2} f_{\mathfrak{t}\mathfrak{r}\mathfrak{t}}(\alpha' - 1)(\xi_{\mathfrak{t}} + A_{\mathfrak{t}}) \approx \frac{1}{2} f_{\mathfrak{t}\mathfrak{r}\mathfrak{t}} x_{\mathfrak{t}} \approx 0$$

$$F_{\mathfrak{r}\mathfrak{r}} \approx i[\xi_{\mathfrak{r}}, \xi_{\mathfrak{r}}] - \frac{1}{2} f_{\mathfrak{r}\mathfrak{r}\mathfrak{t}} \xi_{\mathfrak{t}} = 0$$

---

[11]On scalar fields (i.e. $\phi \in Mat(V_{N\Lambda_2+\Lambda_1})$), it coincides with the usual Killing vector field $\mathcal{L}'_a \phi = [\xi_a, \phi]$ in the large N limit, because $[A_a, f(x)]$ tends to zero as long as we stay away from the "south sphere".

This shows explicitly that F is purely tangential, since $F_{tr} \approx x_t = 0$ at the north pole. This was shown in general in Section 5.2.3. Using the explicit results (243) (which does not require the full calculation in Appendix E), the tangential components are

$$
\begin{aligned}
F_{tt} &\approx f_{tt8} \frac{1}{2\sqrt{3}} \frac{1-2m}{2} \mathbb{1} - \frac{1}{4}(f_{tt3}\sigma_3 + f_{tt1}\sigma_1 + f_{tt2}\sigma_2) \\
&= F_{tt}^{U(1)} \mathbb{1} + F_{tt}^{SU(2)}
\end{aligned}
\tag{188}
$$

at the north pole, which extends to the entire $\mathbb{C}P^2$ since the field strength is invariant under $\mathcal{L}_a$. Comparing with the results of Section 2.2, we see that F_{tt} can be split into a selfdual part proportional to the symplectic form η (this is the monopole part), and an anti-selfdual part; the latter is identified as instanton. Note that there is no m where the field strength is purely selfdual or purely antiselfdual. Therefore this solution is somewhat different from the usual instantons. It is plausible that the $U(1)$ monopole part will play an important role in the coupling to fermions, due to the spin^c structure of $\mathbb{C}P^2$ [30].

Hence we have a gauge field which is finite and well-defined in the commutative limit (apart from a null-set on the south sphere, see below), and its field strength is globally finite. To see that this really describes a connection on some bundle on $\mathbb{C}P^2$, we need to check that the first and second Chern numbers are integer. The corresponding $U(2)$ bundle structure will then be determined. Using

$$
tr_{2\times 2} F_{ab} = (2m-1) F_{ab}^{mono}
\tag{189}
$$

and denoting the basic 2-cocycle with $\omega = \frac{\eta}{3\pi}$, we can read off the first Chern class resp. number as

$$
c_1 = (2m-1)\omega \cong 2m-1.
\tag{190}
$$

As a check, one can also compute c_1 directly from (92), without using the invariance under $SU(3)$. Using

$$
\frac{1}{N} F_a C_a = \frac{-2m+1}{2}
\tag{191}
$$

for large N (see also Section 6.3), we find again (190). Similarly, the 2nd Chern class is

$$
\begin{aligned}
c_2 &= -\frac{1}{8\pi^2}(trF \wedge trF - tr(F \wedge F)) \\
&= -\frac{1}{8\pi^2}\left(trF^{U(1)} \wedge trF^{U(1)} - tr((F^{U(1)} + F^{SU(2)}) \wedge (\star F^{U(1)} - \star F^{SU(2)}))\right) \\
&= -\frac{1}{8\pi^2} tr((F^{U(1)} + F^{SU(2)}) \wedge \star(F^{U(1)} + F^{SU(2)})) \\
&= \frac{1}{2} tr(F_{ab}F_{ab})(\frac{2}{9\pi^2}d^4V) = \frac{1}{2} tr(F_{ab}F_{ab})\,\omega^2
\end{aligned}
\tag{192}
$$

using (86), (89), (101) and and the fact that $F^{SU(2)}$ is anti-selfdual while F^{mono} is selfdual. Curiously, we find that c_2 coincides with the Yang-Mills action, even though F is neither selfdual nor anti-selfdual. This may hint at some underlying BPS structure.

Using (188), we find

$$\frac{1}{2} \, tr_{2\times 2} F_{ab} F_{ab} = 1 - m + m^2. \tag{193}$$

This agrees with the direct calculation of the Yang-Mills action, using (123), as shown in (207). Therefore

$$c_2 = 1 - m + m^2. \tag{194}$$

Note that since $c_1 = 2m - 1$ is obtained by taking the trace over a 2×2 matrix, the individual $U(1)$ charges are half-integer, $m - \frac{1}{2}$. This is consistent with the explicit form of the gauge potential given in (243), which splits into $\frac{1}{2}(-2m + 1)\mathbb{1}_{2\times 2}$ plus a $SU(2)$ part. A similar observation is exploited in [32,33] to construct the charges of the standard model from such nontrivial bundles. This is quite remarkable in view of the common belief that (non-expanded) noncommutative gauge theory allows only the charges $\pm 1, 0$: if we succeed to couple the gauge field to a fermion, this may show a way to circumvent this restriction without expanding in a deformation parameter and using the Seiberg-Witten map [31].

The explicit form of the instanton gauge field can now be worked out using the Gelfand-Tsetlin basis. Because it is somewhat lengthy, we give the result in Appendix E. These field configurations correspond to (generalized) instantons which are "spread out" over the entire $\mathbb{C}P^2$, since $F_{ab}F_{ab} = const$. It would be very interesting to find localized versions of these instantons. In particular, one would like to have "many-instanton" solutions, and a description of the corresponding moduli spaces. Furthermore, it would also be desirable to understand better the connection with the usual self-dual instantons, and / or to construct fuzzy analogs of those for finite N.

6.2.3 The classical bundle structure

Let us compute the total Chern character for the above field configuration:

$$ch = 2 + c_1 + \frac{1}{2}(c_1 \wedge c_1 - 2c_2) = 2 + (2m - 1)\omega + (m^2 - m - \frac{1}{2})\omega^2. \tag{195}$$

As shown in [32,33], this is precisely the Chern character of the bundle

$$L^m \otimes F \tag{196}$$

where L is the generating line bundle (i.e. the above monopole bundle with charge $m = -1$) and F is a nontrivial rank 2 bundle over $\mathbb{C}P^2$ with structure group $U(2)$ defined by $F \oplus L \cong I^3$, where I^3 is the trivial rank 3 bundle. Note that $c_1(F) = -c_1(L)$ and $c_2(F) = c_1(L)^2 \cong 1$.

Therefore our "instanton" gauge field A_a is a connection on the nontrivial $U(2)$ bundle $L^m \otimes F$. It is a solution of $U(2)$ Yang-Mills with finite action, which is neither selfdual

nor anti-selfdual. These bundles are associated to the principal $U(2)$ bundle

$$
\begin{array}{ccc}
U(2) & \longrightarrow & SU(3) \\
& & \downarrow \\
& & \mathbb{C}P^2.
\end{array}
\tag{197}
$$

As explained in [14, 32, 33], a Dirac operator can be defined on this (classical) bundle taking into account the spin^c structure, for spinors transforming under $SU(2) \times U(1)$. All this strongly suggests that it should be possible to define such a Dirac operator also in the fuzzy case in our matrix approach[12], and that physically interesting models including matter could be constructed as matrix models on fuzzy $\mathbb{C}P^2_N$.

We can also see directly how our construction of fuzzy instantons leads in the classical limit to the bundle $L^m \otimes F$. According to the discussion around (122), the (fuzzy version of) sections of the associated $U(2)$ vector bundle on which C acts from the left are given by rectangular matrices $Hom(V_{(0,N)}, V_{(1,M)})$. The harmonic analysis of this space of "fuzzy" sections, i.e. their decomposition as $SU(3)$ module is therefore

$$
\begin{aligned}
V_{(0,N)}^* \otimes V_{(1,M)} &= \bigoplus_{k=0}^{M} (V_{(k,k+m+1)} \oplus V_{(k,k+m-1)}), \qquad m \geq 1 \\
V_{(0,N)}^* \otimes V_{(1,M)} &= \bigoplus_{k=0}^{N} (V_{(k-m,k+1)} \oplus V_{(k-m,k)}), \qquad m < 1
\end{aligned}
\tag{198}
$$

This coincides precisely with the more formal constructions in [14], where it was also shown that it agrees with the space of sections in the bundle[13] $L^m \otimes F$ as $N \to \infty$.

6.3 Solution of the equation of motion

In order to show that (180) contains exact solutions of the equation of motion, we must learn to work with the $\zeta_a = \zeta_a^{(m)}$ operators, which act on V_Λ with $\Lambda = M\Lambda_2 + \Lambda_1$. The following combination turns out to be useful:

$$
\tilde{D} := \tilde{D}^{(m)} = \left(\frac{-M+1}{6} + X\right)^2 - \left(\frac{M+1}{2}\right)^2 = \frac{1}{3}(M+2) + \frac{1}{2}\tilde{D}_c \, \tau^c.
\tag{199}
$$

Here

$$
\tilde{D}_c = d^{ab}{}_c \zeta_a \zeta_b - \frac{1}{3}(2M+7)\zeta_c,
\tag{200}
$$

$X = \zeta_a \tau^a$, and $f\zeta\zeta + 3\zeta = 0$ as well as (202) have been used. This is the combination which occurs in (177). Using (222), it follows that $\tilde{D}$ is a projector on $V_{\Lambda-\Lambda_1}$, and it

---

[12]see [14, 15, 25] for possible definitions of a Dirac operator on fuzzy $\mathbb{C}P^2$

[13]F is denoted with H^{-1} in [14]

vanishes on $V_{\Lambda+\Lambda_2}$ and $V_{\Lambda+\Lambda_2-\alpha_2}$; this is the origin of the above definition. The $\tilde{D}_a$ are clearly covariant, i.e. they transform in the 8 of $su(3)$:

$$[\zeta_a, \tilde{D}_b] = i/2 f_{ab}^c \tilde{D}_c \tag{201}$$

hence they are "tensor operators". Using these properties, One can derive (see Appendix D) the following identities which hold on V_Λ:

$$
\begin{aligned}
\zeta_a \zeta_a &= \frac{1}{3}(M+2)^2 \tag{202} \\
\tilde{D}_a \zeta_a &= -\frac{(M+2)(M+8)}{3} \\
\tilde{D}_a \tilde{D}_a &= \frac{1}{3}(M+2)(10M+32) \\
d_c^{ab} \zeta_a \zeta_b &= \tilde{D}_c + \frac{1}{3}(2M+7)\zeta_c \\
if_c^{ab} \tilde{D}_a \zeta_b &= -3D_c = if_c^{ab}\zeta_a\tilde{D}_b \\
if_c^{ab} \tilde{D}_a \tilde{D}_b &= 4(M+2)\zeta_c + 2(2M+7)\tilde{D}_c \\
d_c^{ab} \tilde{D}_a \zeta_b &= -\frac{4}{3}(2+M)\zeta_c - \frac{1}{3}(2M+7)\tilde{D}_c = d_c^{ab}\zeta_a\tilde{D}_b \\
d_c^{ab} \tilde{D}_a \tilde{D}_b &= -4(M+2)\zeta_c + \frac{2}{3}(2M+7)\tilde{D}_c. \tag{203}
\end{aligned}
$$

It is then easy to see that the Ansatz

$$C_a = \alpha\zeta_a + \beta\tilde{D}_a \tag{204}$$

contains an exact solution of the equation of motion (125) for finite N with α, β as specified in (178). This is so because using the above identities, the equation of motion (125) takes the form

$$r(\alpha,\beta)t_a + s(\alpha,\beta)\tilde{D}_a = 0 \tag{205}$$

for certain functions r, s which depend on the coefficients $\mu_{1,2}$ in the action. To find the solutions to $r = s = 0$ we just have to minimize the positive definite action for C of the form (204), which is possible. Indeed one can easily see that for

$$\alpha = 1 + \frac{m}{N} + o(1/N^2), \qquad \beta = \frac{1}{2N} + o(1/N^2); \tag{206}$$

the constraint $D_a = o(\frac{1}{N})\zeta_a + o(\frac{1}{N})\tilde{D}_a = o(1)$ and also $C_aC_a - (\frac{N^2}{3}+N) = o(1)$ as $N \to \infty$. This is sufficient to guarantee that the gauge fields are tangential according to (112), and that the constraint term vanishes $S_D \to 0$ for $\mu_{1,2} = o(1/N)$. Plugging this into (123) gives the Yang-Mills action,

$$S_{YM} = \frac{1}{g}\int tr(1-m+m^2) \quad + o(1/N) \tag{207}$$

39

in complete agreement with the direct computation (193). Here tr is the trace over the nonabelian matrix content, i.e. $tr_{2\times 2}$ for the $U(2)$ case. Similarly one verifies (191).

One can furthermore show that the constraint $D_a = 0$ does have exact solutions for the above Ansatz (204) consistent with (206), but there exists no solution for $fCC \propto C$ and $dCC \propto C$ simultaneously. This leads to the problem that while the Ansatz is compatible with the constraint $C^2 = (\frac{N+1}{2})^2$ discussed in Section 5.5, it is no longer a solution of the corresponding equations of motion. This is the reason why we concentrate on the formulation with auxiliary variables.

7   Discussion and outlook

We give in this paper a matrix-model formulation of gauge theory on fuzzy $\mathbb{C}P^2$. Our action differs from related matrix models in the context of string theory [18,19] by adding Casimir-type constraint terms following [11], which are designed so as to stabilize the fuzzy $\mathbb{C}P^2$. From a field theoretic point of view, they give the non-tangential degrees of freedom a large mass. This ensures that the usual Yang-Mills action is reproduced in the commutative (large N) limit. We then proceed to find nontrivial solutions of the equation of motion, which turn out to be $U(1)$ monopoles and certain $U(2)$ instanton-like solutions.

The main merits of these models are that the quantization is well-defined and finite, and that topologically nontrivial configurations arise simply as solutions of the matrix equations of motion. In particular, we do not have to sum over disconnected topological sectors; they are included in the "path" integral over all matrices. Unlike in 2 dimensions [11], it would be too much to expect that the model can be solved analytically. However, one may hope that the formulation as matrix model will give a new handle on 4-dimensional gauge theory. We find one interesting simplification at a particular (if unphysical) point in parameter space (156), which might be interesting to pursue.

There are many open issued which deserve further investigations. One is the inclusion of fermions, which is nontrivial due to the fact that $\mathbb{C}P^2$ has no spin but spin^c structure. There are several papers where this is investigated [14, 15, 25], but the appropriate coupling to a gauge field in our formulation is not clear. Another open problem is to find "localized" instantons and their moduli space. This is complicated by the apparent lack of a Hodge-star (with correct classical limit) on $\mathbb{C}P_N^2$. In particular, our instantons contain a nontrivial $U(1)$ sector, and are neither selfdual nor anti-selfdual. The $U(1)$ monopole part seems to be related to the spin^c structure on $\mathbb{C}P^2$, and may be important for the coupling to fermions [30].

We also give an alternative formulation by imposing the constraint (136) rather than using auxiliary fields. The relation of these different formulations is not clear, in particular we have not found instanton solutions in the constrained case. Finally, it would be very desirable to get some insight into the large N behavior of the quantized model. This could

be studied using renormalization group techniques developed in [28].

Acknowledgements

Appendix A: Explicit form of the $su(3)$ generators

Denote the simple roots of $su(3)$ with α_1, α_2, and the highest root with $\theta = \alpha_1 + \alpha_2$. The corresponding Cartan-Weyl generators satisfy the usual relations

$$
\begin{aligned}
[X_i^+, X_j^-] &= \delta_{i,j} H_{\alpha_i}, \qquad i = 1, 2 \\
[X_1^+, X_2^+] &= X_\theta^+
\end{aligned}
$$

etc. Furthermore, $[X_\theta^+, X_2^-] = X_1^+$, $[X_2^+, X_\theta^-] = X_1^-$. It is also useful to define

$$
\begin{aligned}
H_3 &= H_{\alpha_1}, \\
H_\theta &= H_{\alpha_1} + H_{\alpha_2}, \\
\sqrt{3}\, H_8 &= H_{\alpha_1} + 2 H_{\alpha_2}.
\end{aligned}
\tag{208}
$$

This provides a normalized basis T_a, $a = 1, 2, ..., 8$ of $su(3)$ defined by

$$
\begin{aligned}
T_4 \pm i T_5 &= X_\theta^\pm, \qquad T_6 \pm i T_7 = X_2^\pm, \qquad T_1 \pm i T_2 = X_1^\pm, \\
T_3 &= \frac{1}{2} H_3, \qquad T_8 = \frac{1}{2} H_8.
\end{aligned}
\tag{209}
$$

The usual Gell-Mann matrices are then given by

$$
\lambda_a = 2\pi_{\Lambda_1}(T_a)
\tag{210}
$$

where π_{Λ_1} denotes the defining representation V_{Λ_1} with weights $\nu_i = (\Lambda_1, \Lambda_1 - \alpha_1, \Lambda_1 - \alpha_1 - \alpha_2 = -\Lambda_2)$. However, for our purpose it is convenient to use the "conjugated" Gell-mann matrices, defined by

$$
\tau_a = 2\pi_{\Lambda_2}(T_a).
\tag{211}
$$

Here the weights of the dual representation V_{Λ_2} are $\nu_i = (\Lambda_2, \Lambda_2 - \alpha_2, -\Lambda_1)$. They are explicitly given by

$$
\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad
\tau_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad
\tau_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

$$\tau_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \tau_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$

$$\tau_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \tau_7 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tau_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{212}$$

One can now compute the explicit form of f_{abc} and d_{abc} defined by (5):

$$\begin{aligned}
f_{12}{}^3 &= 2, \ f_{14}{}^7 = 1, \ f_{15}{}^6 = -1, \ f_{24}{}^6 = 1, \ f_{25}{}^7 = 1, \\
f_{34}{}^5 &= 1, \ f_{36}{}^7 = -1, \ f_{45}{}^8 = \sqrt{3}, \ f_{67}{}^8 = \sqrt{3},
\end{aligned} \tag{213}$$

and

$$\begin{aligned}
d_{11}{}^8 &= -2/\sqrt{3}, \ d_{14}{}^6 = -1, \ d_{15}{}^7 = -1, \ d_{22}{}^8 = -2/\sqrt{3}, \ d_{24}{}^7 = 1, \ d_{25}{}^6 = -1 \\
d_{33}{}^8 &= -2/\sqrt{3}, \ d_{34}{}^4 = -1, \ d_{35}{}^5 = -1, \ d_{36}{}^6 = 1, \ d_{37}{}^7 = 1, d_{44}{}^8 = 1/\sqrt{3}, \\
d_{55}{}^8 &= 1/\sqrt{3}, \ d_{66}{}^8 = 1/\sqrt{3}, \ d_{77}{}^8 = 1/\sqrt{3}, \ d_{88}{}^8 = 2/\sqrt{3}.
\end{aligned} \tag{214}$$

Note also that

$$\sum_{a,b} f_{abc} f_{abd} = 12 \delta_{cd}. \tag{215}$$

Appendix B: characteristic equation

Consider

$$X = \sum_a t_a \tau^a = \frac{1}{2} ((t_a + \tau_a)^2 - t_a t^a - \tau_a \tau^a) \tag{216}$$

and recall the eigenvalues of the quadratic Casimirs on the highest weight representation $V_{N_1 \Lambda_1 + N_2 \Lambda_2}$ are given by

$$c_2(\Lambda) = (\Lambda, \Lambda + 2\rho) = \frac{2}{3}(N_1^2 + N_2^2 + N_1 N_2) + 2(N_1 + N_2) \tag{217}$$

where $\rho = \Lambda_1 + \Lambda_2$ is the Weyl vector of $su(3)$. We used here

$$(\Lambda_1, \Lambda_1) = \frac{2}{3} = (\Lambda_2, \Lambda_2), \quad (\Lambda_1, \Lambda_2) = \frac{1}{3}, \quad (\alpha_1, \alpha_1) = 2. \tag{218}$$

Now

$$V_\Lambda \otimes V_{\Lambda_2} = V_{\Lambda + \nu_1} \oplus V_{\Lambda + \nu_2} \oplus V_{\Lambda + \nu_3} \tag{219}$$

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where $\nu_i = \Lambda_2, \Lambda_2 - \alpha_2, -\Lambda_1$ are the weights of the fundamental representation V_{Λ_2}. If $N_1 = 0$, then the last summand does not occur. This implies that

$$X = \frac{1}{2}(c_2(\Lambda + \nu) - c_2(\Lambda) - c_2(\Lambda_2)) = (\nu, \Lambda + \rho) - (\Lambda_2, \rho) \tag{220}$$

on $V_{\Lambda+\nu}$, which gives

$$X = (\frac{2N}{3} + \frac{n}{3}, \ -\frac{N}{3} + \frac{n}{3} - 1, \ -\frac{N}{3} - \frac{2n}{3} - 2\ ) \tag{221}$$

on $V_{\Lambda+\nu_i}$ for $i = 1, 2, 3$ and $\Lambda = N\Lambda_2 + n\Lambda_1$. Hence the characteristic equation of X is

$$(X - \frac{2N}{3} - \frac{n}{3})(X + \frac{N}{3} - \frac{n}{3} + 1)(X + \frac{N}{3} + \frac{2n}{3} + 2) = 0 \tag{222}$$

as long as $n > 0$, while for $n = 0$ the last factor disappears and

$$(X - \frac{2N}{3})(X + \frac{N}{3} + 1) = 0. \tag{223}$$

Appendix C: rewriting the Yang-Mills action

The following identity holds for $su(3)$ (see e.g. [34])

$$\sum_e d_{abe}d_{cde} = \frac{1}{3}\left(4(\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \sum_e(f_{ace}f_{bde} + f_{ade}f_{bce})\right). \tag{224}$$

Using the Jacobi-identity

$$\sum_e f_{ade}f_{bce} + f_{cae}f_{bde} + f_{abe}f_{cde} = 0 \tag{225}$$

we get

$$\sum_e d_{abe}d_{cde} = \frac{1}{3}\left(4(\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}) + \sum_e(2f_{ade}f_{bce} + f_{abe}f_{cde})\right). \tag{226}$$

Contracting this with $C_aC_bC_cC_d$ gives

$$Tr\left(-(dCC)(dCC) + \frac{1}{3}\left(4\delta_{ac}\delta_{bd} + \sum_e(2(fCC)_eC_dC_af_{ade} + (fCC)(fCC))\right)\right) = 0 \tag{227}$$

hence

$$Tr\left(\frac{1}{2}[C,C]^2 + (C \cdot C)(C \cdot C)\right) = Tr\left(\frac{3}{4}(dCC)(dCC) + \frac{1}{4}\sum_e(fCC)(fCC)\right) \tag{228}$$

where $\frac{1}{2}[C,C]^2 = C_aC_bC_aC_b - (C \cdot C)(C \cdot C)$.

Appendix D: Covariant operators for instantons

We must learn to work with the $\zeta = \zeta_a^{(m)}$ operators, which act on V_Λ with $\Lambda = M\Lambda_2 + \Lambda_1$. To handle them, it is useful to consider

$$\tilde{D} := \tilde{D}^{(m)} = (\frac{-M+1}{6} + X)^2 - (\frac{M+1}{2})^2 = \frac{M+2}{3} + \frac{1}{2}\tilde{D}_a^{(m)}\tau^a \qquad (229)$$

where $X = \zeta_a\tau^a$ and (200)

$$\tilde{D}_c^{(m)} = d^{ab}{}_c\zeta_a\zeta_b - \frac{1}{3}(2M+7)\zeta_c.$$

Using (222), it follows that $\tilde{D}$ satisfies the equation

$$\tilde{D}X = -\frac{M+8}{3}\,\tilde{D}. \qquad (230)$$

It is a projector on $V_{\Lambda - \Lambda_1}$, and vanishes on $V_{\Lambda + \Lambda_2}$ and $V_{\Lambda + \Lambda_2 - \alpha_2}$. This implies

$$\tilde{D}_a\zeta_a = \zeta_a\tilde{D}_a = tr(\tilde{D}X) = -\frac{M+8}{3}\,tr(\tilde{D}) = -\frac{(M+2)(M+8)}{3}. \qquad (231)$$

Similarly, from considering $tr(\tilde{D}\tilde{D})$ one obtains

$$\tilde{D}_a\tilde{D}_a = \frac{1}{3}(M+2)(10M+32), \qquad (232)$$

which shows that $\tilde{D}_a$ is an operator of order N. We also note the Casimir

$$\zeta_a\zeta_a = \frac{1}{2}(\Lambda, \Lambda + 2\rho) = \frac{(M+2)^2}{3}. \qquad (233)$$

Now consider $if_c^{ab}\tilde{D}_a\zeta_b$. It is a covariant operator on V_Λ in the sense of (201). However by a suitable generalization of the Wigner-Eckart theorem[14] there are only two such operators, which must be ζ_a and $\tilde{D}_a$. Therefore

$$if_c^{ab}\tilde{D}_a\zeta_b = \alpha\zeta_c + \beta\tilde{D}_c \qquad (234)$$

for some constants α, β. Contracting with ζ_c, we get

$$if^{abc}\tilde{D}_a\zeta_b\zeta_c = \alpha\zeta_c\zeta_c + \beta\tilde{D}_c\zeta_c = -3\tilde{D}_a\zeta_a = (M+2)(M+8).$$

Therefore

$$(3+\beta)(M+8) = \alpha(M+2).$$

---

[14]i.e. by decomposing the intertwiner space $V_\Lambda \otimes V_\Lambda^*$

Similarly,

$$i f^{abc} \tilde{D}_a [\zeta_b, \tilde{D}_c] = \frac{i}{2} i f^{abc} f^d_{bc} \tilde{D}_a \tilde{D}_d = -6 \tilde{D}_a \tilde{D}_a = 2(\alpha \zeta_c + \beta \tilde{D}_c) \tilde{D}_c \qquad (235)$$

using (215) implies

$$(3 + \beta)(10M + 32) = \alpha(M + 8).$$

Solving these gives

$$\boxed{i f^{ab}_c \tilde{D}_a \zeta_b = -3 \tilde{D}_c = i f^{ab}_c \zeta_a \tilde{D}_b} \qquad (236)$$

since ζ_a is selfadjoint.

Now consider again $\tilde{D}X = -\frac{M+8}{3} \tilde{D}$. Writing it out in components gives

$$\frac{1}{4}(if + d)_{abc} \tilde{D}_a \zeta_b + \frac{M+2}{3} \zeta_c = -\frac{M+8}{6} \tilde{D}_c \qquad (237)$$

which using the above result gives

$$\boxed{d^{ab}_c \tilde{D}_a \zeta_b = -\frac{4}{3}(2 + M)\zeta_c - \frac{1}{3}(2M + 7)\tilde{D}_c = d^{ab}_c \zeta_a \tilde{D}_b} \qquad$$

Next, consider

$$d^{ab}_c \tilde{D}_a \tilde{D}_b = \alpha' \zeta_c + \beta' \tilde{D}_c. \qquad (238)$$

Contracting with ζ_c and using the previous results gives

$$(4(2 + M) + 3\beta')(M + 8) - (2M + 7)(10M + 32) = \alpha' 3(M + 2)$$

To proceed, we need some extra information on $\tilde{D}_a$. We claim that on the "bottom" of the representation V_Λ (see figure 1 for illustration), i.e. at the lowest eigenvalue $-\frac{1}{2\sqrt{3}}(M+2)$ of ζ_8, we have $\tilde{D}_{1,2,3} = 0$ and $\tilde{D}_8 = \frac{2}{\sqrt{3}}(M+2)$. One way to see this is to note that $\tilde{D}_{1,2,3} = x\zeta_{1,2,3}$ there by covariance, since the multiplicity is 1 on the "bottom"; then (231),(232),(233) together imply $x = 0$. It can also be seen using the projector property of $\tilde{D}$. The eigenvalue of $\tilde{D}_8$ can then be calculated explicitly using the quadratic Casimir:

$$\tilde{D}_8 = d^{ab}{}_8 \zeta_a \zeta_b - \alpha \zeta_8 = \frac{1}{\sqrt{3}} \left(\left(\sum_a \zeta_a \zeta_a\right) - 3(\zeta_1 \zeta_1 + \zeta_2 \zeta_2 + \zeta_3 \zeta_3) + \zeta_8 \zeta_8 \right) - \frac{1}{3}(2M + 7)\zeta_8, \qquad (239)$$

which gives $\tilde{D}_8 = \frac{2}{\sqrt{3}}(M+2)$ on the bottom of V_Λ. We can now calculate

$$
\begin{aligned}
d_{8ab} \tilde{D}_a \tilde{D}_b &= d_{8aa} \tilde{D}_a \tilde{D}_a = \frac{1}{\sqrt{3}} \left(\left(\sum_a \tilde{D}_a \tilde{D}_a\right) - 3(\tilde{D}_1 \tilde{D}_1 + \tilde{D}_2 \tilde{D}_2 + \tilde{D}_3 \tilde{D}_3) + \tilde{D}_8 \tilde{D}_8 \right) \\
&= \frac{1}{\sqrt{3}} \left(\left(\sum_a \tilde{D}_a \tilde{D}_a\right) + \tilde{D}_8 \tilde{D}_8 \right) \\
&= \frac{2}{3\sqrt{3}}(M + 2)(7M + 20) = (M + 2)(-\alpha' \frac{1}{2\sqrt{3}} + \beta' \frac{2}{\sqrt{3}}) \qquad (240)
\end{aligned}
$$

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Therefore
$$2(7M + 20) = -\frac{3}{2}\alpha' + 6\beta'.$$

Together with the above this gives $\alpha' = -4(M + 2)$, $\beta' = \frac{2}{3}(2M + 7)$ and

$$\boxed{d_c^{ab}\tilde{D}_a\tilde{D}_b = -4(M + 2)\zeta_c + \frac{2}{3}(2M + 7)\tilde{D}_c}$$

Together with $\tilde{D}^2 = (2M + 6)\tilde{D}$ written in components this implies

$$\boxed{if_c^{ab}\tilde{D}_a\tilde{D}_b = 4(M + 2)\zeta_c + 2(2M + 7)\tilde{D}_c}$$

Appendix E: Explicit form of the instanton field

We first calculate the field
$$A_a'(x) = \alpha'\zeta_a^{(m)} - \xi_a \tag{241}$$
for the above configuration of $U(2)$ gauge theory explicitly. We have to choose a gauge, i.e. a nice embedding of $V_{N\Lambda_2} \oplus V_{N\Lambda_2} \subset V_{M\Lambda_2+\Lambda_1}$. More precisely, we must find a suitable basis for these two spaces to define the embedding of $\xi_a \otimes \mathbb{1}$, and calculate the difference. Again we use the Gelfand-Tsetlin basis for $V_{M\Lambda_2+\Lambda_1}$ and $V_{N\Lambda_2} \oplus V_{N\Lambda_2}$ (see Appendix F), where the operators can be calculated explicitly. One finds (263)

$$
\begin{aligned}
2\sqrt{3}\zeta_8^{(m)} &= 2\sqrt{3}\xi_8 + (-2m + 1), \quad 2\zeta_3^{(m)} = 2\xi_3 + \sigma^3, \\
(\zeta_4 \pm i\zeta_5)^{(m)} &= (\xi_4 \pm i\xi_5)^{(M+\epsilon)}, \\
(\zeta_6 - i\zeta_7)^{(m)} &= (\xi_6 - i\xi_7)^{(M+\epsilon)}(1 - h(x)\frac{\sigma^3}{2M}) + \frac{x_4 - ix_5}{\sqrt{3}}h(x)\frac{\sqrt{3}}{\sqrt{1 + 2x_8}}\frac{1}{2}\sigma_+ \\
&= (\xi_6 - i\xi_7)^{(M+\epsilon)} - \frac{x_6 - ix_7}{\sqrt{3}}h(x)\frac{1}{2}\sigma_3 + \frac{x_4 - ix_5}{\sqrt{3}}h(x)\frac{\sqrt{3}}{\sqrt{1 + 2x_8}}\frac{1}{2}\sigma_+ \\
(\zeta_6 + i\zeta_7)^{(m)} &= (\xi_6 + i\xi_7)^{(M+\epsilon)}(1 - h(x)\frac{\sigma^3}{2M}) + \frac{x_4 + ix_5}{\sqrt{3}}h(x)\frac{\sqrt{3}}{\sqrt{1 + 2x_8}}\frac{1}{2}\sigma_- \\
&= (\xi_6 + i\xi_7)^{(M+\epsilon)} - \frac{x_6 + ix_7}{\sqrt{3}}h(x)\frac{1}{2}\sigma_3 + \frac{x_4 + ix_5}{\sqrt{3}}h(x)\frac{\sqrt{3}}{\sqrt{1 + 2x_8}}\frac{1}{2}\sigma_- \\
(\zeta_1 \pm i\zeta_2)^{(m)} &= (1 - h(x)\frac{\sigma^3}{2M})(\xi_1 \pm i\xi_2) + h(x)\frac{\sqrt{1 + 2x_8}}{\sqrt{3}}\frac{1}{2}\sigma^\pm \\
&= (\xi_1 \pm i\xi_2) - \frac{x_1 + ix_2}{\sqrt{3}}h(x)\frac{1}{2}\sigma_3 + h(x)\frac{\sqrt{1 + 2x_8}}{\sqrt{3}}\frac{1}{2}\sigma^\pm \tag{242}
\end{aligned}
$$

where (261)
$$h(x) = \frac{3}{2 + x_8 - \sqrt{3}x_3}$$

and $\sigma_\pm = \sigma_1 \pm i\sigma_2$. Using $\alpha' = 1 + \frac{2m-1}{2N} + o(1/N^2)$, this gives

$$A'_8 = \frac{2m-1}{2} A_8^{mono}, \quad A'_3 = \frac{2m-1}{2} A_3^{mono} + \frac{1}{2}\sigma^3,$$

$$A'_4 \pm iA'_5 = \frac{2m-1}{2}(A_4^{mono} \pm iA_5^{mono}) + \frac{\sqrt{3}}{2}\frac{x_4 \pm ix_5}{2x_8+1}\frac{1}{2}\sigma^3$$

$$A'_6 \pm iA'_7 = \frac{2m-1}{2}(A_6^{mono} \pm iA_7^{mono}) + \frac{\sqrt{3}}{2}\frac{x_6 \pm ix_7}{2x_8+1}\frac{1}{2}\sigma^3$$

$$- \frac{x_6 \pm ix_7}{\sqrt{3}} h(x)\frac{1}{2}\sigma_3 + \frac{x_4 \pm ix_5}{\sqrt{3}} h(x)\frac{\sqrt{3}}{\sqrt{1+2x_8}}\frac{1}{2}\sigma_\mp$$

$$A'_1 \pm iA'_2 = \frac{2m-1}{2}(A_1^{mono} \pm iA_2^{mono}) - h(x)\frac{x_1 \pm ix_2}{\sqrt{3}}\frac{1}{2}\sigma_3 + h(x)\frac{\sqrt{1+2x_8}}{\sqrt{3}}\frac{1}{2}\sigma^\pm$$

$$\tag{243}$$

One can check using (21) that $A_a x_a = 0$, as it must be. At first sight, the $U(1)$ part looks like a half-integer monopole, which is not allowed. However, this can be rewritten as

$$A'_8 = A_8^{mono}Q + \frac{1}{\sqrt{3}}(x_8-1)\frac{1}{2}\sigma_3$$

$$A'_3 = A_3^{mono}Q + \frac{1}{\sqrt{3}}(x_3+\sqrt{3})\frac{1}{2}\sigma^3,$$

$$A'_4 \pm iA'_5 = (A_4^{mono} \pm iA_5^{mono})Q + \frac{1}{\sqrt{3}}(x_4 \pm ix_5)\frac{1}{2}\sigma^3$$

$$A'_6 \pm iA'_7 = (A_6^{mono} \pm iA_7^{mono})Q + \frac{x_6 \pm ix_7}{\sqrt{3}}(1-h(x))\frac{1}{2}\sigma_3 + h(x)\frac{x_4 \pm ix_5}{\sqrt{1+2x_8}}\frac{1}{2}\sigma_\mp$$

$$A'_1 \pm iA'_2 = (A_1^{mono} \pm iA_2^{mono})Q + \frac{x_1 \pm ix_2}{\sqrt{3}}(1-h(x))\frac{1}{2}\sigma_3 + h(x)\frac{\sqrt{1+2x_8}}{\sqrt{3}}\frac{1}{2}\sigma^\pm$$

where the $U(1)$ monopole part is associated to the generator

$$Q = \frac{2m-1}{2} - \frac{1}{2}\sigma_3 = \begin{pmatrix} m-1 & 0 \\ 0 & m \end{pmatrix} \tag{244}$$

which has "charge 1", i.e. $e^{2\pi iQ} = 1$ but $e^{\pi iQ} \neq 1$. This is consistent with the usual quantization condition. We therefore write

$$A'_a = QA_a^{mono} + (A_a^{inst})'. \tag{245}$$

Projecting out the non-tangential part as in (177) is straightforward in the commutative limit, writing $A_a = A'_a + (\frac{1}{\sqrt{3}}d_{abc}x_b A'_c - \frac{1}{3}A'_a)$. Since this map is linear and the monopole part is already tangential, we have

$$A_a = QA_a^{mono} + A_a^{inst} = QA_a^{mono} + (A_a^{inst})' + (\frac{1}{\sqrt{3}}d_{abc}x_b(A_c^{inst})' - \frac{1}{3}(A_a^{inst})').$$

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Explicitly, this is is using (21)

$$A_1^{inst} \pm i A_2^{inst} = \frac{x_1 \pm i x_2}{\sqrt{3}}(2 - h(x))\frac{1}{2}\sigma_3 + \sqrt{\frac{1 + 2x_8}{3}}(h(x) - 1)\frac{1}{2}\sigma_\pm$$

$$A_3^{inst} = (2\frac{x_3}{\sqrt{3}} + 1 - \frac{1 + 2x_8}{3}h(x))\frac{1}{2}\sigma_3 - \frac{\sqrt{2x_8 + 1}}{3}h(x)\frac{1}{2}(x_1\sigma_1 + x_2\sigma_2)$$

$$A_4^{inst} = \frac{h(x)}{\sqrt{1 + 2x_8}^3}\left(x_4 x_5(x_7\sigma_1 + x_6\sigma_2) + \frac{1}{6}(3x_4^2 - 3x_5^2 - (1 + 2x_8)^2)(x_6\sigma_1 - x_7\sigma_2)\right)$$

$$- \frac{h(x)}{\sqrt{3}(1 + 2x_8)}(x_6^2 + x_7^2)x_4\sigma_3$$

$$A_5^{inst} = \frac{h(x)}{\sqrt{1 + 2x_8}^3}\left(x_4 x_5(-x_6\sigma_1 - x_7\sigma_2) + \frac{1}{6}(3x_4^2 - 3x_5^2 - (1 + 2x_8)^2)(x_7\sigma_1 - x_6\sigma_2)\right)$$

$$- \frac{h(x)}{\sqrt{3}(1 + 2x_8)}(x_6^2 + x_7^2)x_5\sigma_3$$

$$A_6^{inst} \pm i A_7^{inst} = \frac{h(x)}{2\sqrt{1+2x_8}^3}(x_6^2 + x_7^2)(x_4 \pm i x_5)(\sigma_1 \mp i\sigma_2) + \frac{1}{2\sqrt{3}}(2 - h(x))(x_6 \pm i x_7)\sigma_3$$

$$A_8^{inst} = \left(-\frac{1}{3}x_3 + \frac{1}{6\sqrt{3}}(-2 + (h(x) - 1)(2x_1^2 + 2x_2^2 - x_6^2 - x_7^2) - 2x_3^2 + x_4^2 + x_5^2 + 2x_8^2)\right)\sigma_3$$

$$+ \frac{h(x)}{6\sqrt{1+2x_8}}\left((x_4 x_6 + x_5 x_7 - \frac{2}{\sqrt{3}}(1 + 2x_8)x_1)\sigma_1 + (x_5 x_6 - x_4 x_7 - \frac{2}{\sqrt{3}}(1 + 2x_8)x_1)\sigma_2\right)$$

which can be seen to be tangential.

Appendix F: Gelfand-Tsetlin basis for monopoles and instantons

We need the explicit form for the action of the Lie algebra on $V_{N\Lambda_2 + n\Lambda_1}$. This is given in terms of the Gelfand-Tsetlin basis (see [35] for a review), which works as follows. Consider complexified $gl(3) \cong su(3) \oplus u(1)$, with generators being the elementary matrices E_{ij} and the Cartan subalgebra $\mathfrak{h}$ generated by E_{ii}. Irreps of $gl(3)$ are highest weight reps V_μ with highest weights being labeled by 3 complex numbers $\mu = (\mu_1, \mu_2, \mu_3)$ such that

$$\mu_1 \geq \mu_2 \geq \mu_3. \tag{246}$$

The highest weight vector satisfies $E_{ii} |\mu\rangle = \mu_i|\mu\rangle$ for $i = 1, 2, 3$, and $E_{ij} |\mu\rangle = 0$ for $1 \leq i < j \leq 3$. The relation with the $su(3)$ weights is as follows:

$$\Lambda_1 \cong (1, 0, 0), \qquad \Lambda_2 \cong (1, 1, 0) \tag{247}$$

where the overall sum is the $u(1)$ charge and is ignored for $su(3)$. The relations among the generators are

$$H_{\alpha_1} = E_{11} - E_{22} = H_3, \qquad H_{\alpha_2} = E_{22} - E_{33},$$

$$\sqrt{3}\, H_8 \;=\; H_{\alpha_1} + 2H_{\alpha_2} = E_{11} + E_{22} - 2E_{33},$$
$$X_1^+ \;=\; E_{1,2}, \qquad X_2^+ = E_{2,3},$$
$$X_1^- \;=\; E_{2,1}, \qquad X_2^- = E_{3,2},$$

One then associates to such a weight μ the pattern

$$\mathcal{P} = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ & p \quad q & \\ & r & \end{pmatrix} = \begin{pmatrix} \mu_{1,3} & \mu_{2,3} & \mu_{3,3} \\ & \mu_{1,2} \quad \mu_{2,2} & \\ & \mu_{1,1} & \end{pmatrix} \tag{248}$$

where

$$\mu_1 \;\geq\; p \geq \mu_2 \geq q \geq \mu_3,$$
$$p \;\geq\; r \geq q$$

One can then show that a basis of V_μ is given by the orthonormal weight vectors

$$|\mathcal{P}\rangle \equiv |\mu; p, q, r\rangle \equiv |\mathcal{P}_\mu\rangle,$$

and the action of the $gl(3)$ generators is

$$
\begin{aligned}
E_{1,1}\,|\mathcal{P}\rangle &= r\,|\mathcal{P}\rangle, \\
E_{2,2}\,|\mathcal{P}\rangle &= (p + q - r)\,|\mathcal{P}\rangle, \\
E_{3,3}\,|\mathcal{P}\rangle &= (\mu_1 + \mu_2 + \mu_3 - p - q)\,|\mathcal{P}\rangle, \\
H_{\alpha_1}\,|\mathcal{P}\rangle &= (2r - p - q)\,|\mathcal{P}\rangle, \\
H_{\alpha_2}\,|\mathcal{P}\rangle &= (2(p + q) - r - (\mu_1 + \mu_2 + \mu_3))\,|\mathcal{P}\rangle, \\
E_{1,2}\,|\mathcal{P}\rangle &= A_1^1(\mathcal{P})\,|\mathcal{P}_{r+1}\rangle, \\
E_{2,1}\,|\mathcal{P}\rangle &= A_1^1(\mathcal{P}_{r-1})\,|\mathcal{P}_{r-1}\rangle, \\
E_{2,3}\,|\mathcal{P}\rangle &= A_2^1(\mathcal{P})\,|\mathcal{P}_{p+1}\rangle + A_2^2(\mathcal{P})\,|\mathcal{P}_{q+1}\rangle, \\
E_{3,2}\,|\mathcal{P}\rangle &= A_2^1(\mathcal{P}_{p-1})\,|\mathcal{P}_{p-1}\rangle + A_2^2(\mathcal{P}_{q-1})\,|\mathcal{P}_{q-1}\rangle
\end{aligned}
$$

where $|\mathcal{P}_{p-1}\rangle$ means that p is replaced by $p-1$, etc, and it is supposed that $|\mathcal{P}\rangle = 0$ if $\mathcal{P}$ is not a pattern. Here

$$
\begin{aligned}
A_1^1(\mathcal{P}) &= \left(-(\mu_{1,2} - \mu_{1,1})(\mu_{2,2} - \mu_{1,1} - 1)\right)^{1/2}, \\
A_2^1(\mathcal{P}) &= \left(-\frac{(\mu_{1,3} - \mu_{1,2})(\mu_{2,3} - \mu_{1,2} - 1)(\mu_{3,3} - \mu_{1,2} - 2)(\mu_{1,1} - \mu_{1,2} - 1)}{(\mu_{2,2} - \mu_{1,2} - 1)(\mu_{2,2} - \mu_{1,2} - 2)}\right)^{1/2}, \\
A_2^2(\mathcal{P}) &= \left(-\frac{(\mu_{1,3} - \mu_{2,2} + 1)(\mu_{2,3} - \mu_{2,2})(\mu_{3,3} - \mu_{2,2} - 1)(\mu_{1,1} - \mu_{2,2})}{(\mu_{1,2} - \mu_{2,2} + 1)(\mu_{1,2} - \mu_{2,2})}\right)^{1/2}. \tag{249}
\end{aligned}
$$

Consider first $N\Lambda_2 \cong (N, N, 0)$. The associated pattern is

$$\mathcal{P} = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ & p \quad q & \\ & r & \end{pmatrix} = \begin{pmatrix} \mu_{1,3} & \mu_{2,3} & \mu_{3,3} \\ & \mu_{1,2} \quad \mu_{2,2} & \\ & \mu_{1,1} & \end{pmatrix} = \begin{pmatrix} N & N & 0 \\ & N \quad q & \\ & r & \end{pmatrix} \tag{250}$$

hence $N \geq r \geq q \geq 0$, and the highest weight state is $q = r = N$. Then

$$\begin{aligned}
A_1^1(\mathcal{P}) &= ((N-r)(r+1-q))^{1/2}, \\
A_2^1(\mathcal{P}) &= 0, \\
A_2^2(\mathcal{P}) &= ((q+1)(r-q))^{1/2}.
\end{aligned} \tag{251}$$

Hence

$$\begin{aligned}
H_{\alpha_1} |\mathcal{P}\rangle &= (2r - q - N) |\mathcal{P}\rangle, \\
H_{\alpha_2} |\mathcal{P}\rangle &= (2q - r) |\mathcal{P}\rangle, \\
\sqrt{3} H_8 |\mathcal{P}\rangle &= (3q - N) |\mathcal{P}\rangle = H_{\alpha_1} + 2H_{\alpha_2} |\mathcal{P}\rangle, \\
X_1^+ |\mathcal{P}\rangle &= \sqrt{(N-r)(r+1-q)} |\mathcal{P}_{r+1}\rangle, \\
X_1^- |\mathcal{P}\rangle &= \sqrt{(N-r+1)(r-q)} |\mathcal{P}_{r-1}\rangle, \\
X_2^+ |\mathcal{P}\rangle &= \sqrt{(q+1)(r-q)} |\mathcal{P}_{q+1}\rangle, \\
X_2^- |\mathcal{P}\rangle &= \sqrt{q(r-q+1)} |\mathcal{P}_{q-1}\rangle \\
X_\theta^+ |\mathcal{P}\rangle &= -\sqrt{(q+1)(N-r)} |\mathcal{P}_{r+1,q+1}\rangle \\
X_\theta^- |\mathcal{P}\rangle &= -\sqrt{q(N-r+1)} |\mathcal{P}_{r-1,q-1}\rangle
\end{aligned}$$

using $X_\theta^+ = [X_1^+, X_2^+]$.

It turns out that this basis is not useful to calculate the instanton fields (it would lead to a singularity at the north pole). A better basis is obtained by applying a Weyl rotation S_2 along the root α_2, defining

$$|\mathcal{P}\rangle' = S_2|\mathcal{P}\rangle \tag{252}$$

The corresponding automorphism $T_2(X) = S_2 X S_2^{-1}$ is

$$\begin{aligned}
T_2(H_2) &= -H_2, & T_2(H_1) &= H_1 + H_2 \\
T_2(X_2^\pm) &= -X_2^\mp, & T_2(X_1^\pm) &= -X_\theta^\pm, & T_2(X_\theta^\pm) &= X_1^\pm.
\end{aligned}$$

Hence

$$\begin{aligned}
H_{\alpha_1} |\mathcal{P}\rangle' &= (r + q - N) |\mathcal{P}\rangle' & &= (-a + b) |\mathcal{P}\rangle', \\
\sqrt{3} H_8 |\mathcal{P}\rangle' &= (3r - 3q - N) |\mathcal{P}\rangle' & &= (2N - 3(a+b)) |\mathcal{P}\rangle', \\
X_\theta^+ |\mathcal{P}\rangle' &= -\sqrt{(N-r)(r+1-q)} |\mathcal{P}_{r+1}\rangle' & &= -\sqrt{a(N-a-b+1)} |\mathcal{P}_{a-1}\rangle', \\
X_\theta^- |\mathcal{P}\rangle' &= -\sqrt{(N-r+1)(r-q)} |\mathcal{P}_{r-1}\rangle' & &= -\sqrt{(a+1)(N-a-b)} |\mathcal{P}_{a+1}\rangle', \\
X_2^- |\mathcal{P}\rangle' &= -\sqrt{(q+1)(r-q)} |\mathcal{P}_{q+1}\rangle' & &= -\sqrt{(b+1)(N-a-b)} |\mathcal{P}_{b+1}\rangle', \\
X_2^+ |\mathcal{P}\rangle' &= -\sqrt{q(r-q+1)} |\mathcal{P}_{q-1}\rangle' & &= -\sqrt{b(N-a-b+1)} |\mathcal{P}_{b-1}\rangle', \\
X_1^+ |\mathcal{P}\rangle' &= -\sqrt{(q+1)(N-r)} |\mathcal{P}_{r+1,q+1}\rangle' & &= -\sqrt{a(b+1)} |\mathcal{P}_{a-1,b+1}\rangle', \\
X_1^- |\mathcal{P}\rangle' &= -\sqrt{q(N-r+1)} |\mathcal{P}_{r-1,q-1}\rangle' & &= -\sqrt{(a+1)b} |\mathcal{P}_{a+1,b-1}\rangle'
\end{aligned} \tag{253}$$

introducing the parameters $a = N - r, b = q$ which range from $0 \leq a, b \leq N, 0 \leq a + b \leq N$. The north pole (with maximal eigenvalue of H_8) satisfies $r = N, q = 0$ and $\sqrt{3} H_8 = 2N$, or $a = b = 0$. The parameters a, b measure the deviation from the north pole.

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To find the monopole gauge field, we have to repeat the same calculation replacing N by $M = N - m$. Since we want to match the north poles (i.e. $A = 0$ at the north pole), we can simply match the a and b parameters. This defines a map $V_{M\Lambda_2} \mapsto V_{N\Lambda_2}$ except at the south sphere, and the generators of $V_{M\Lambda_2}$ can be expressed in terms of the generators of $V_{N\Lambda_2}$ as

$$
\begin{aligned}
H_{\alpha_1}^{(M)} &= H_{\alpha_1}, \\
\sqrt{3}H_8^{(M)} &= \sqrt{3}H_8 + 2(M - N), \\
X_1^{\pm(M)} &= X_1^{\pm}, \\
X_2^{\pm(M)} &= X_2^{\pm}(1 + \frac{3}{2}\frac{M - N}{\sqrt{3}H_8 + N} + o(1/N^2)), \\
X_\theta^{\pm(M)} &= X_\theta^{\pm}(1 + \frac{3}{2}\frac{M - N}{\sqrt{3}H_8 + N} + o(1/N^2)).
\end{aligned}
\tag{254}
$$

Using

$$
\xi_8 = T_8 = \frac{1}{2}H_8, \quad \xi_3 = T_3 = \frac{1}{2}H_3 \quad \xi_1 \pm i\xi_2 = X_1^{\pm}, \quad \xi_4 \pm i\xi_5 = X_\theta^{\pm}, \quad \xi_6 \pm i\xi_7 = X_2^{\pm},
\tag{255}
$$

this leads to (172).

Instantons

Next consider $N\Lambda_2 + n\Lambda_1 \cong (n + N, N, 0)$. The associated pattern is

$$
\mathcal{P} = \begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ & p & q \\ & & r \end{pmatrix} = \begin{pmatrix} \mu_{1,3} & \mu_{2,3} & \mu_{3,3} \\ & \mu_{1,2} & \mu_{2,2} \\ & & \mu_{1,1} \end{pmatrix} = \begin{pmatrix} N+n & N & 0 \\ & N+\epsilon & q \\ & & r \end{pmatrix}
\tag{256}
$$

where $n \geq \epsilon \geq 0$ and $N + \epsilon \geq r \geq q \geq 0$ and $N \geq q$. We only need $n = 1$, so that $\epsilon = 0, 1$. Then the highest weight state is given by $\epsilon = 1, q = N, r = N + 1$. Furthermore $A_2^1(\mathcal{P}) = 0$ for $\epsilon = 1$, and

$$
\begin{aligned}
A_1^1(\mathcal{P}) &= \sqrt{(N + \epsilon - r)(r + 1 - q)}, \\
A_2^1(\mathcal{P}) &= \delta_{\epsilon,0}\sqrt{N - r + 1}\left(\frac{(N + 2)}{(N - q + 1)(N - q + 2)}\right)^{1/2}, \\
A_2^2(\mathcal{P}) &= \sqrt{(q + 1)(r - q)}\left(\frac{(N + 2 - q)(N - q)}{(N + \epsilon - q + 1)(N + \epsilon - q)}\right)^{1/2}.
\end{aligned}
\tag{257}
$$

Note that since $\epsilon = 0, 1$ only, this can be written as

$$
\left(\frac{(N + 2 - q)(N - q)}{(N + \epsilon - q + 1)(N + \epsilon - q)}\right)^{1/2} = \sqrt{1 - \frac{\sigma_3}{N + 1 - q}}
\tag{258}
$$

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where $\sigma_3 = 2\epsilon - 1$. We also define

$$h(q) = \sqrt{\frac{N(N+2)}{(N-q+1)(N-q+2)}}. \tag{259}$$

Then

$$
\begin{aligned}
H_{\alpha_1} |\mathcal{P}\rangle &= (2r - q - \epsilon - N) |\mathcal{P}\rangle, \\
H_{\alpha_2} |\mathcal{P}\rangle &= (2q + 2\epsilon - r - 1) |\mathcal{P}\rangle, \\
\sqrt{3} H_8 |\mathcal{P}\rangle &= (3q + 3\epsilon - 2 - N) |\mathcal{P}\rangle, \\
X_1^+ |\mathcal{P}\rangle &= \sqrt{(N + \epsilon - r)(r + 1 - q)} \, |\mathcal{P}_{r+1}\rangle, \\
X_1^- |\mathcal{P}\rangle &= \sqrt{(N + \epsilon - r + 1)(r - q)} \, |\mathcal{P}_{r-1}\rangle, \\
X_2^+ |\mathcal{P}\rangle &= \delta_{\epsilon,0} \sqrt{(N - r + 1)/N} h(q)|\mathcal{P}_{p+1}\rangle \\
&\quad + \sqrt{(q+1)(r-q)} \sqrt{1 - \frac{\sigma_3}{N + 1 - q}} \, |\mathcal{P}_{q+1}\rangle, \\
X_2^- |\mathcal{P}\rangle &= \delta_{\epsilon,1} \sqrt{(N - r + 1)/N} h(q) \, |\mathcal{P}_{p-1}\rangle \\
&\quad + \sqrt{q(r - q + 1)} \sqrt{1 - \frac{\sigma_3}{N + 2 - q}} \, |\mathcal{P}_{q-1}\rangle \\
X_\theta^+ |\mathcal{P}\rangle &= \delta_{\epsilon,0} \sqrt{(r + 1 - q)/N} h(q)|\mathcal{P}_{r+1,p+1}\rangle \\
&\quad - \sqrt{(q+1)(N + \epsilon - r)} \sqrt{1 - \frac{\sigma_3}{N + 1 - q}} |\mathcal{P}_{r+1,q+1}\rangle \\
X_\theta^- |\mathcal{P}\rangle &= \delta_{\epsilon,1} \sqrt{(r - q)/N} h(q)|\mathcal{P}_{r-1,p-1}\rangle \\
&\quad - \sqrt{q(N + \epsilon - r + 1)} \sqrt{1 - \frac{\sigma_3}{N + 2 - q}} |\mathcal{P}_{r-1,q-1}\rangle
\end{aligned}
$$

We should find that this is approximately $\xi_a \otimes \mathbb{1}_2$. This is not natural in this basis (the singularity would be at the north pole). A more suitable basis is found after a Weyl reflection $|\mathcal{P}\rangle' = S_2|\mathcal{P}\rangle$, as in the last section. The north pole then satisfies $q = 0$, and either $\epsilon = 0, r = N$ or $\epsilon = 1, r = N + 1$ with $\sqrt{3} H_8 = N = 2N$ as it should be.

Figure 1 shows schematically the decomposition of $V_{N\Lambda_2 + \Lambda_1}$ into the eigenspaces of $\epsilon = 0, 1$. In terms of the parameters $a = N + \epsilon - r, b = q$ with the range $0 \le a \le N + \epsilon$, $0 \le b \le N$, $a + b \le N + \epsilon$ we get

$$
\begin{aligned}
H_{\alpha_1} |\mathcal{P}\rangle' &= (b - a + 2\epsilon - 1) |\mathcal{P}\rangle', \\
\sqrt{3} H_8 |\mathcal{P}\rangle' &= (2N - 3(a + b) + 1) |\mathcal{P}\rangle', \\
X_\theta^+ |\mathcal{P}\rangle' &= -\sqrt{a(N - a - b + \epsilon + 1)} \, |\mathcal{P}_{a-1}\rangle', \\
X_\theta^- |\mathcal{P}\rangle' &= -\sqrt{(a + 1)(N - a - b + \epsilon)} \, |\mathcal{P}_{a+1}\rangle', \\
X_2^- |\mathcal{P}\rangle' &= -\delta_{\epsilon,0} \sqrt{\tfrac{a+1}{N}} h(b)|\mathcal{P}_{\epsilon+1,a+1}\rangle' - \sqrt{(b + 1)(N + \epsilon - a - b)} \sqrt{1 - \tfrac{\sigma_3}{N+1-b}} \, |\mathcal{P}_{b+1}\rangle',
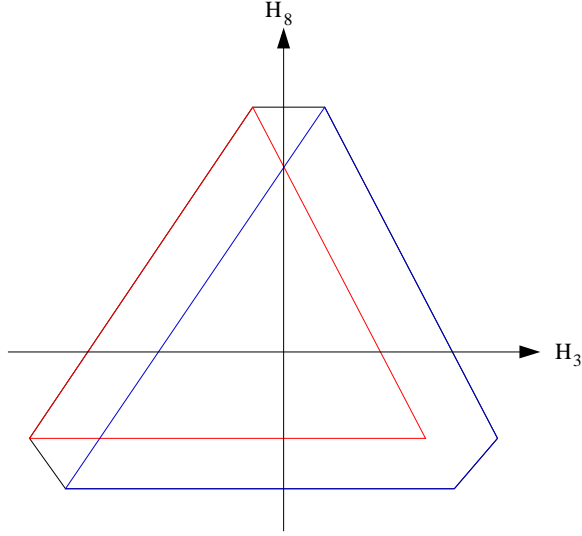\end{aligned}
$$

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Figure 1: decomposition of $V_{N\Lambda_2+\Lambda_1}$ into the eigenspaces of $\epsilon = 0, \epsilon = 1$

$$X_2^+ \, |\mathcal{P}\rangle' \;=\; -\delta_{\epsilon,1} \sqrt{\frac{a}{N}} h(b) \, |\mathcal{P}_{\epsilon-1,a-1}\rangle' - \sqrt{b(N+\epsilon-a-b+1)} \sqrt{1 - \frac{\sigma_3}{N+2-b}} \, |\mathcal{P}_{b-1}\rangle'$$

$$X_1^+ \, |\mathcal{P}\rangle' \;=\; \delta_{\epsilon,0} \sqrt{\frac{N-a-b+1}{N}} h(b) |\mathcal{P}_{\epsilon=1}\rangle' - \sqrt{(b+1)a} \sqrt{1 - \frac{\sigma_3}{N+1-b}} |\mathcal{P}_{a-1,b+1}\rangle'$$

$$X_1^- \, |\mathcal{P}\rangle' \;=\; \delta_{\epsilon,1} \sqrt{\frac{N+1-a-b}{N}} h(b) |\mathcal{P}_{\epsilon=0}\rangle' - \sqrt{b(a+1)} \sqrt{1 - \frac{\sigma_3}{N+2-b}} |\mathcal{P}_{a+1,b-1}\rangle'$$

It is now obvious that the representation splits naturally into two subspaces with $\epsilon = 0$ and $\epsilon = 1$, each of which can be mapped to $V_{N\Lambda_2}$ by matching the parameters a, b with the corresponding basis of the previous section. Looking at the bounds for a and b, this works perfectly as long as $a + b \leq N$. Hence we have a map

$$V_{N\Lambda_2+\Lambda_1} \mapsto V_{N\Lambda_2} \otimes \mathbb{C}^2 = \begin{pmatrix} \mathcal{P}_{\epsilon=1} \\ \mathcal{P}_{\epsilon=0} \end{pmatrix}$$

except for the lowest eigenvalue of H_8, which is the south sphere. This is the "coordinate patch" of $\mathbb{C}P^2$ where the instanton gauge field will be well-defined. Using this identification, we can express the above generators in terms of the generators for $V_{N\Lambda_2}$ by comparing with (253), i.e. by writing the maps on $V_{N\Lambda_2} \otimes \mathbb{C}^2$ in the form

$$C_a = C_{a,\alpha} \sigma^\alpha \tag{260}$$

where $\sigma_3 = 2\epsilon - 1$ etc. For large N, we can write

$$1 - \frac{b}{N} = \frac{1}{3}(2 + x_8 - \sqrt{3}x_3), \qquad 1 - \frac{a+b}{N} = \frac{1}{3}(1 + 2x_8)$$

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so that using (255) and (253) we have e.g.

$$-\sqrt{(a+1)/N}h(b)|\mathcal{P}_{\epsilon+1,a+1}\rangle' \;=\; \frac{\xi_4-i\xi_5}{N}\sqrt{\frac{3}{1+2x_8}}h(x)\,|\mathcal{P}_{\epsilon+1,a}\rangle'$$

$$-\sqrt{a/N}h(b)|\mathcal{P}_{\epsilon-1,a-1}\rangle' \;=\; \frac{\xi_4+i\xi_5}{N}\sqrt{\frac{3}{1+2x_8}}h(x)\,|\mathcal{P}_{\epsilon+1,a}\rangle'$$

where

$$h(x)=\sqrt{\frac{N(N+2)}{(N-b+1)(N-b+2)}}=\frac{3}{2+x_8-\sqrt{3}x_3} \tag{261}$$

note also

$$\sqrt{\frac{(N-a-b+1)(N+2)}{(N-b+1)(N-b+2)}} \;=\; \sqrt{\frac{1+2x_8}{3}}h(x),$$

$$\sqrt{1-\frac{\sigma_3}{N+2-b}} \;=\; \sqrt{1-\frac{h(x)}{N}\sigma_3}. \tag{262}$$

Then the above operators written in this notation are as follows:

$$\sqrt{3}H_8 \;=\; (2\sqrt{3}\xi_8+1)\mathbb{1}\,,$$

$$H_{\alpha_1} \;=\; 2\xi_3+\sigma^3,$$

$$X_\theta^+ \;=\; \begin{pmatrix} (\xi_4+i\xi_5)^{(N+1)} & 0 \\ 0 & (\xi_4+i\xi_5)^{(N)} \end{pmatrix},$$

$$X_\theta^- \;=\; \begin{pmatrix} (\xi_4-i\xi_5)^{(N+1)} & 0 \\ 0 & (\xi_4-i\xi_5)^{(N)} \end{pmatrix},$$

$$X_2^- \;=\; \begin{pmatrix} (\xi_6-i\xi_7)^{(N+1)} & 0 \\ 0 & (\xi_6-i\xi_7)^{(N)} \end{pmatrix}\sqrt{1-\frac{h(x)}{N}\sigma_3}$$

$$+\frac{\xi_4-i\xi_5}{N}\sqrt{\frac{3}{1+2x_8}}h(x)\,\frac{1}{2}\sigma^+,$$

$$X_2^+ \;=\; \begin{pmatrix} (\xi_6+i\xi_7)^{(N+1)} & 0 \\ 0 & (\xi_6+i\xi_7)^{(N)} \end{pmatrix}\sqrt{1-\frac{h(x)}{N}\sigma_3}$$

$$+\frac{\xi_4+i\xi_5}{N}\sqrt{\frac{3}{1+2x_8}}h(x)\,\frac{1}{2}\sigma^-,$$

$$X_1^+ \;=\; (\xi_1+i\xi_2)\sqrt{1-\frac{h(x)}{N}\sigma_3}+\sqrt{\frac{1+2x_8}{3}}h(x)\frac{1}{2}\sigma^+,$$

$$X_1^- \;=\; (\xi_1-i\xi_2)\sqrt{1-\frac{h(x)}{N}\sigma_3}+\sqrt{\frac{1+2x_8}{3}}h(x)\frac{1}{2}\sigma^-,$$

using $\frac{1}{2}\sigma^+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ etc. We can now expand

$$\sqrt{1-\frac{h(x)}{N}\sigma_3}=1-\frac{h(x)}{2N}\sigma_3+o(1/N^2) \tag{263}$$

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which for large N leads to (242).

Note that the "south sphere" is given by $a + b = max = N + 1$ and belongs to $\epsilon = 1$.

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